iii) If beam is axially rigid.

$$D_k = 3j - R_e + \Sigma r - m''$$
$$= 3 \times 10 - 6 + 0 - 5$$
$$= 19$$

Influence Line Diagram (ILD)

ILD represents variation of stress function (Reaction, shear force, bending moment, slope, deflection etc.) at a particular point on the structure due to the movement of unit concentrated load from one end to another end. It gives only qualitative ILD's

Application of ILD :-

1) ILD can be used to study the effect of a moving load on the structure

2) ILD can be used to find the position of live load corresponding to maximum value of a particular stress function

3) ILD can be used to calculate total value of a particular stress function due to a given unit load system.

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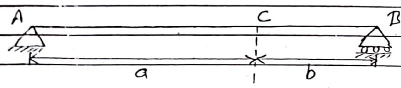
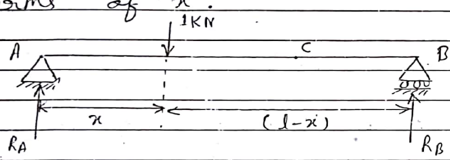
...w the ILD for R_A, R_B and Shear force at 'C' and 'Bending Moment' at 'C' using '' principle.



ILD of R_A →

Consider unit concentrated load at distance 'x' from support A & find the value of R_A in terms of x.



$$\Sigma M_B = 0$$

$$R_A \cdot l - 1(l-x) = 0$$

$$R_A = \frac{l-x}{l}$$

$$R_A = 1 - \frac{x}{l}$$

By applying boundary condition & join the boundary points.

at $x = 0$, $R_A = 1$

at $x = l$, $R_A = 0$

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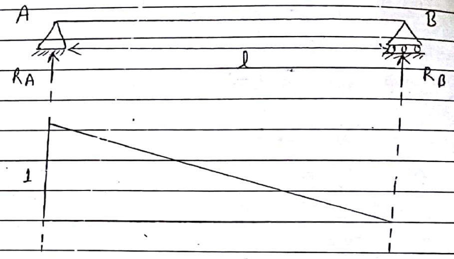


ILD for R_A

ILD for R_B →

$$\Sigma F_y = 0$$

$$R_A + R_B - 1 = 0$$

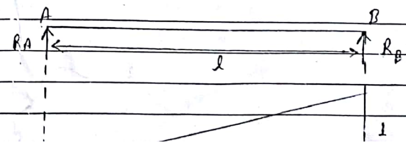
$$1 - \frac{x}{l} + R_B - 1 = 0$$

$$R_B = \frac{x}{l}$$

Apply boundary condition

at $x = 0$, $R_B = 0$

at $x = l$, $R_B = 1$



ILD for S_c →

(i) When $x < a$

$$S_c = R_A - 1$$

$$= 1 - \frac{x}{l} - 1$$

$$S_c = -\frac{x}{l}$$

Apply boundary condition →

at $x = 0$, $S_c = 0$

at $x = a$, $S_c = -\frac{a}{l}$

(ii) when $l > x \geq a$ →

$$S_c = R_A$$

$$S_c = 1 - \frac{x}{l}$$

Apply boundary condition →

at $x = 0$, $S_c = 1$

at $x = a$, $S_c = 1 - \frac{a}{l} = \frac{b}{l}$

at $x = l$, $S_c = 1 - \frac{l}{l} = 0$

$1 + 25 = 26$

$4 + 36 = 40$

$16 + 0 = 16$

$1 + 36 = 37$

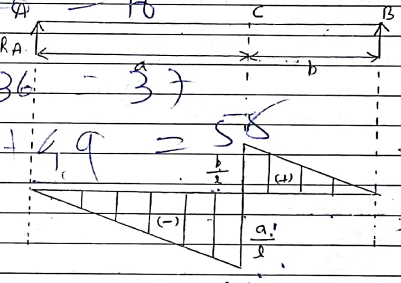
$9 + 49 = 58$

$25 + 64$

$\Rightarrow 89$

$\Rightarrow 64 + 81$

$\Rightarrow 145$



ILD for S.F.$_c$

ILD for Bending Moment at 'C' →

1) when $x \leq a$ →

$$B.M_c = R_B \cdot b$$

$$B.M_c = \frac{x}{l} \cdot b$$

Apply boundary condition →

at $x = 0$, $B.M_c = 0$

at $x = a$, $B.M_c = \frac{ab}{l}$

ii) when $a \leq x \leq l$:

$$B.M_c = R_A \cdot a$$

$$B.M_c = (1 - x) \cdot a$$

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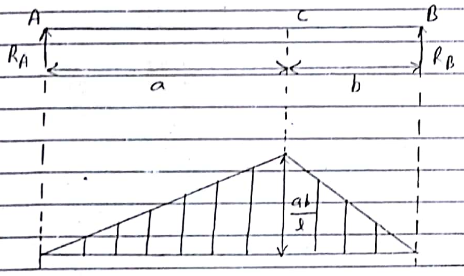
Apply boundary condition →

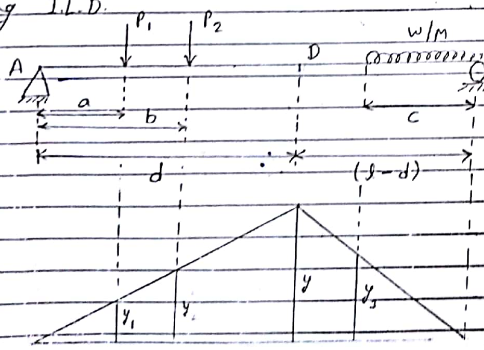
at $x = a$, $B.M_c = (1 - \frac{a}{l}) \cdot a = \frac{ab}{l}$

at $x = l$, $B.M_c = \frac{ab}{l}$ 0



ILD for B.M_c

Q 2.) Due to given loading find B.M at D using I.L.D.



ILD for BM_D

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$y = \frac{ab}{l}$

$y = \frac{d \times (l - d)}{l}$

By using similar triangle method →

$\frac{y_1}{y} = \frac{a}{d}$

$y_1 = \frac{a}{d} \cdot y$

$y_2 = \frac{b}{d} \cdot y$

$y_3 = \frac{c}{(l-d)} \cdot y$

then $B.M_D = P_1 y_1 + P_2 y_2 +$ load × area

$wc \times \frac{y_3}{2}$

or $w \left[\frac{1}{2} y_3 \cdot c \right]$

$B.M_D = P_1 y_1 + P_2 y_2 + w \times$ Area under ILD

2) Muller Breslau Principle →

→ The ILD of any stress function in a structure is represented by deflected shape of the structure obtained

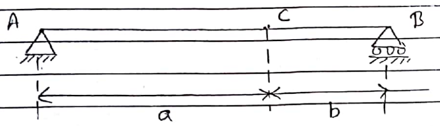
i) Removing the restrain offered by the stress function and

ii) Introducing a directly related

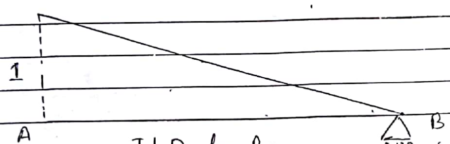
generlised unit displacement in direction of that stress function.

It gives only qualitative ILD's. It means we cannot calculate ordinates always specially in indeterminate str.

→ Using Muller Breslau principle (MBP), Draw ILD for R_A, R_B, S_c & $B.M_c$ in the given structure.

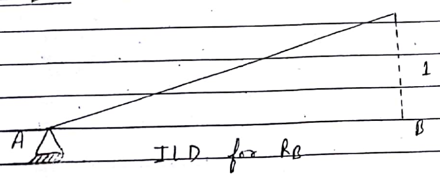


ILD for R_A →



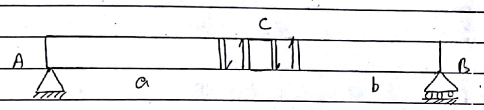
ILD for R_A

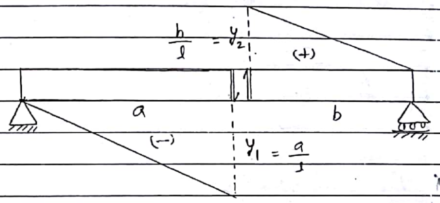
ILD for R_B →



ILD for R_B

ILD for S_c →





$\frac{b}{l} = y_2$

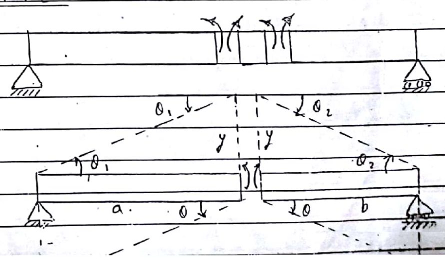
$y_1 = \frac{a}{l}$

$\because \; y_1 + y_2 = 1$

Extend the ILD upto the support B where $y_2 = 0$ & $y_1 = 1$, so using similar triangle get the value of y_1 and y_2.

$y_1 = \frac{a}{l}$ and $y_2 = \frac{b}{l}$

ILD for $B.M_c$ →



$\because \quad \theta = \theta_1 + \theta_2 = 1$

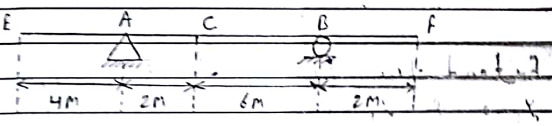
$\because \quad \theta_1$ and θ_2 are very small.

then $\theta = \tan\theta_1 + \tan\theta_2 = 1$

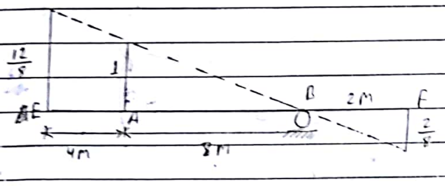
$$\frac{y}{a} + \frac{y}{b} = 1$$

$$y = \frac{ab}{l}$$

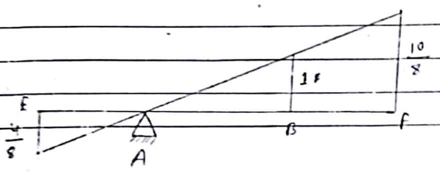
Example-1 Draw the ILD for R_A, R_B, S_c & $B.M_c$ using MBP in the given structure.



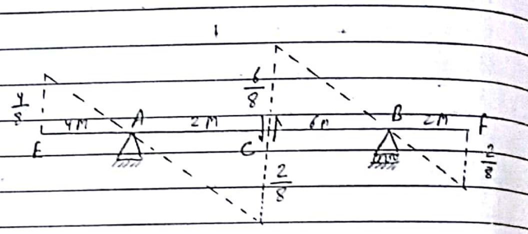
Sol. ILD for R_A →



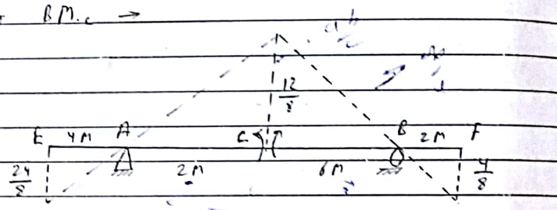
ILD for R_B →



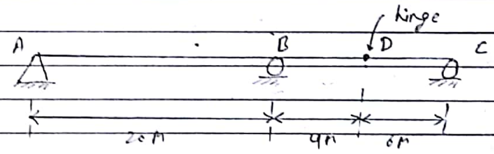
ILD for S_c →



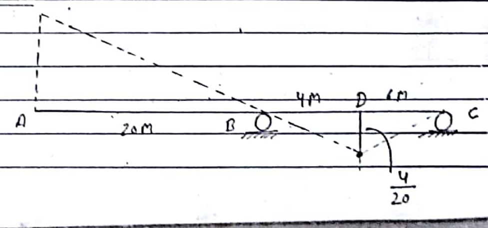
ILD for BM_c →



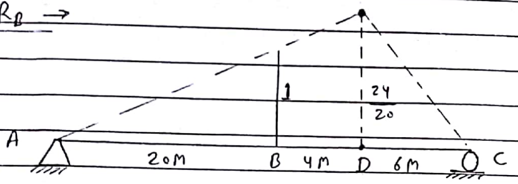
Example-2 Draw the ILD for R_A, R_B, & M_D using MBP in the given structure.



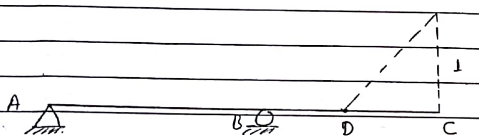
IDL for R_A →



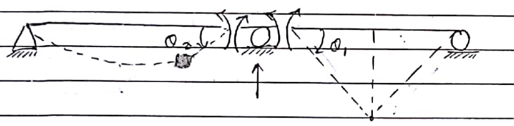
for R_B →



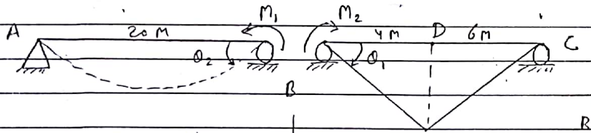
for R_C →



for M_B →



$\therefore \theta_1 + \theta_2 = 1$



$\theta_2 = \dfrac{Ml}{3EI}$

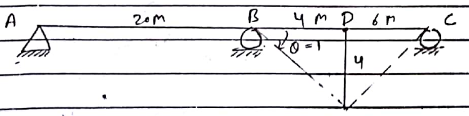
$M_1 = \dfrac{3EI}{l} \cdot \theta_2$

$\Sigma M_D = 0$

$M_2 - R_C \times 10 = 0$

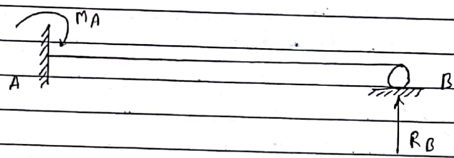
$M_2 = 0$

$M_D = 0$
$R_c \times 6 = 0$
$R_c = 0$

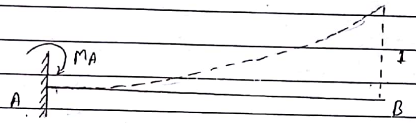


ILD for M_B

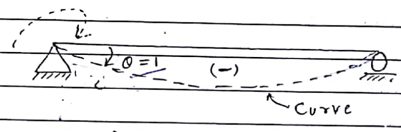
Example - 3 Draw the ILD for R_B and M_A.



Sol ILD for R_B →



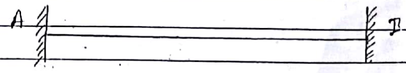
ILD for M_A →



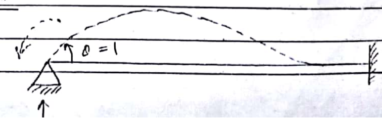
Curve = f(x)

*** ILD's for determinate structure are linear while ILD's for indeterminate structure are curve.

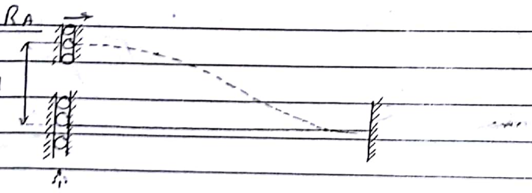
Example - 4 Draw ILD for M_A & R_A, M_B, R_B



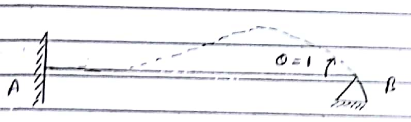
ILD for M_A →



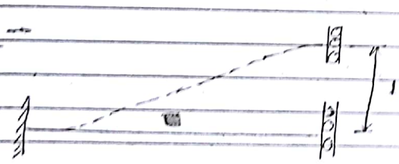
ILD for R_A →



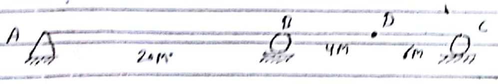
ILD for M_B →



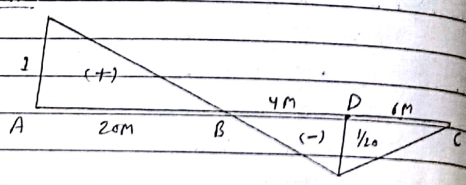
ILD for R_B →



Example 5 Referring the Q-2 find the max. values of above stress functions when a live load of 10 KN/m moves from left to right which may have any length, so as to produce max value of the above stress function.



ILD for R_A →



If UDL is in span AB →

R_A = W × Area under UDL in IL

$= 10 \times \frac{1}{2} \times 20 \times 1$

$= 100$ KN

If UDL is in span BC →

$R_A = 10 \times \left(-\frac{1}{2} \times 10 \times \frac{4}{20}\right)$

$= -10$ KN

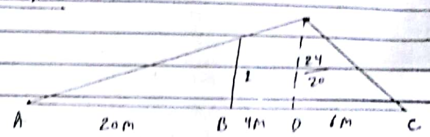
If whole span is loaded →

$R_A = 100 - 10$

$= 90$ KN

For Maximum R_A UDL should be in span AB

ILD for R_B →

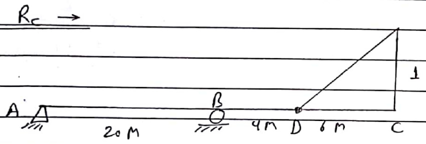


For Maximum R_B, whole should be loaded

$R_{B(max)} = 10 \times \frac{1}{2} \times 30 \times \frac{24}{20}$

$= 180$ KN.

ILD for R_C →

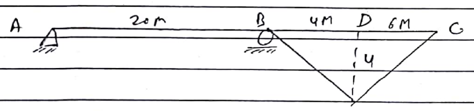


For Maximum R_C at lest span DC should be loaded.

$R_{C(max)} = 10 \times \frac{1}{2} \times 6$

$= 30$ KN

ILD for M_B →

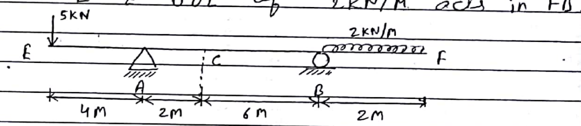


For $M_{B\,max}$ at lest span BC should be loaded

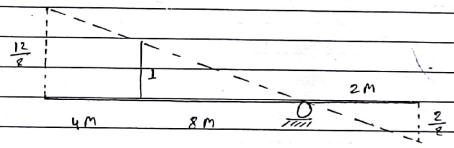
$M_{B(max)} = 10 \times \left(-\frac{1}{2} \times 10 \times 4^2\right)$

$= -200$ KN-M

Example-6 Referring to Q-1, find values of above stress function when a concentrated load of 5KN acts at E & UDL of 2KN/M acts in FB.

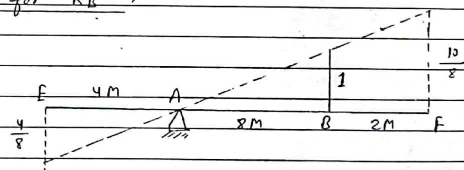


Sol → ∵ ILD for R_A →



then $R_A = 5 \times \frac{12}{8} - 2 \times \frac{1}{2} \times 2 \times \frac{2}{8} \approx 7$ KN

∵ ILD for R_B →



then $R_B = -5 \times \frac{4}{8} + 2 \times \left[\frac{1}{2} \times 2 \times 0.25 + 1 \times 2\right]$

$= 2$ KN

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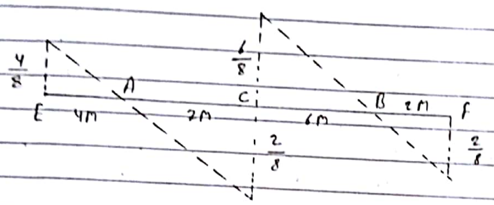
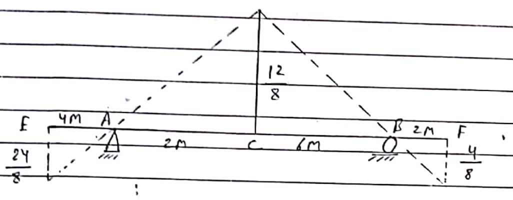
∵ ILD for S_c →



then $S_c = 5 \times \frac{4}{8} - \frac{1}{2} \times 2 \times \frac{2}{8} \times 2$

$= 2\,KN$

∵ ILD for BM_c →



then $BM_c = -5 \times \frac{24}{8} - \frac{1}{2} \times 2 \times \frac{4}{8} \times 2$

$= -16\,KN\text{-}m$

→ if length of span < length of loading then for Max. B.M u[illegible] whole span length
→ and if length of span > length of loading then for Max B.M the moving UDL should be divided by the section in the same ratio ~~in which section divides the span.~~

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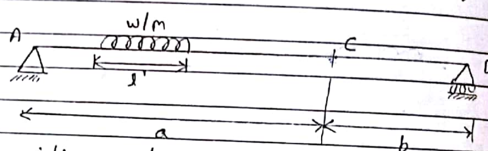
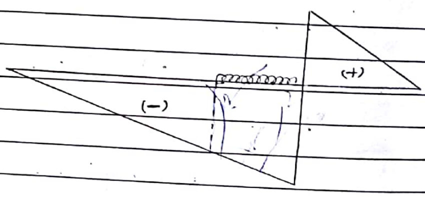
Effect of Rolling Loads →

Q find maximum shear force and ben[ding] moment at section C when a U[DL] passes over a girder from left to [right] having length less than girder len[gth] also find the position of this U[DL] maximum shear force and bending mo[ment]



(i) find position of UDL for maximum SF_c

sol



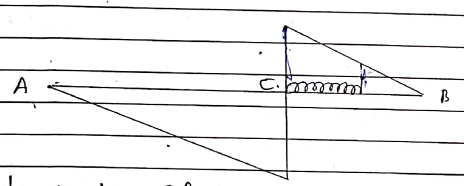
$S_c = W \times$ Area under UDL in ILD.

For $S_{c\,(max)} = W \times A_{max}$

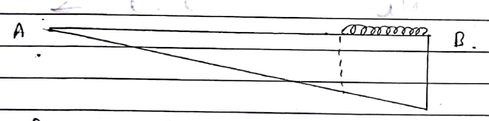
for A_{max}, UDL should be placed near the section 'C'.

For maximum $-ve\ S_c \rightarrow$
UDL should be placed left of C i.e head of UDL should be at C.

For maximum $+ve\ S_c \rightarrow$
UDL should be placed right of C i.e tail of UDL should be at C.



For Absolute maximum S.F. $\rightarrow$
For $(\text{Shear force})_{\text{absolute max.}} = w \times (A)_{\text{absolute max.}}$



Shear force will be maximum near the support and UDL should be placed near the support.

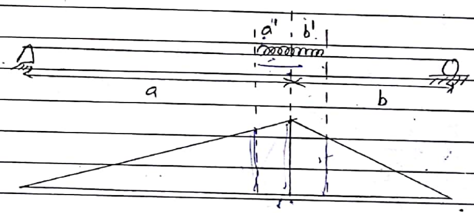
ii) Find position of UDL for maximum $(BM_c) \rightarrow$

$B.M_c = w \times A$

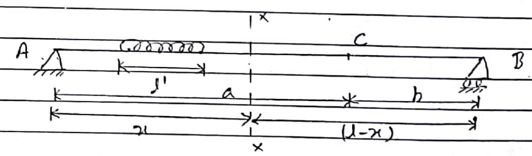
$(B.M_c)_{max.} = w \times A_{max.}$

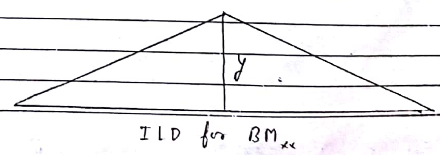
$$\frac{a'}{b'} = \frac{a}{b}$$

Maximum B.M. at C will occur when average loading to the left of C is equal to average loading to the right of C. It means section C will divide the UDL in the same ratio in which it divides the span.



iii) Find the position of UDL for absolute max. B.M which may occur any where.





ILD for BM_{xx}

$$y = \frac{x(l-x)}{l}$$

For Max $y \rightarrow \frac{dy}{dx} = 0$

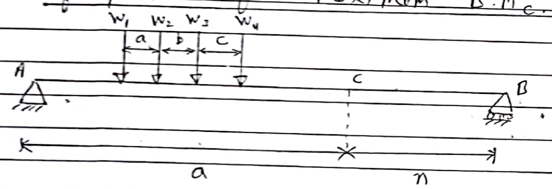
$$\frac{l-2x}{l} = 0$$

$$x = \frac{l}{2}$$

then for Max y section should be at centre.

For absolute maximum B.M. UDL should be placed symmetrical at the centre.

Case-2 (a) Find the position of a system of point for maximum B.M$_c$.

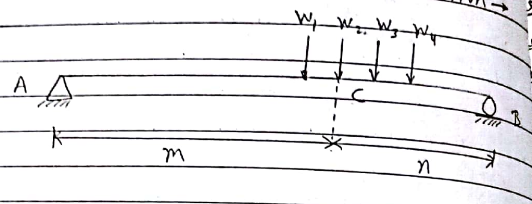


For maximum B.M$_c$ system should be fulfill two criterias →

i) Any one of the load should be at c

ii) Average loading to left of section should be close to avg loading to right of section.

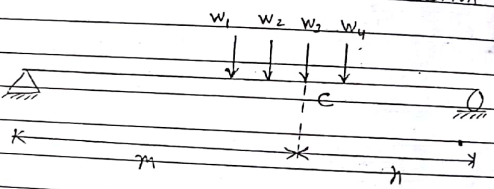
Assume W_2 load at the section →



Average loading to left of section =

" " " right " " = W_3

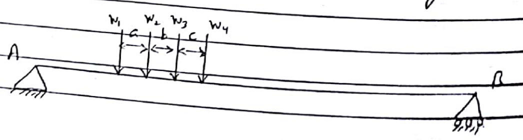
Assume W_3 load at the section →



Avg loading to left of section $= \frac{W_1+W_2}{m}$

" " " right " " $= \frac{W_4}{n}$

Case 2(b) → Find the position of a system of point loads for absolute maximum B.M which may occur anywhere on gir

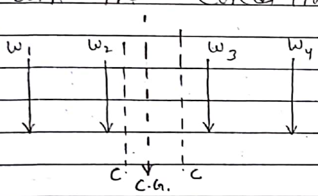


bsolute max. B.M occurs ~~at~~ below one of the wheel loads and "position of load such that centre of the span is midway b/w C.G of load system and load under consideration".

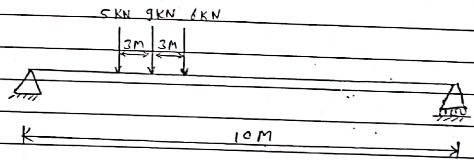
Load under consideration is that load which is either nearest to the C.G. of the load system or next nearest to C.G. of load system. But heavier then nearest load to C.G.

1. If nearest load to C.G is heavier, then M_{max} will occur below this load. And centre of the span will be midway b/w C.G and this load.

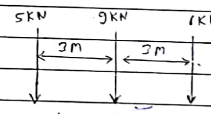
2. If next nearest load to C.G is heavier then nearest load to C.G then M_{max} may occur below any of these two loads, check both the condition.



Q. A series of three wheel loads 5, 9, 6 ton spaced 3M c/c crosses a girder simply support at end with span of 10M. Loads are moving left to right with 6 ton load leading. Find position of this system for absolute maximum B.M. which may occur any where on the girder. Also find the value of M_{max}.



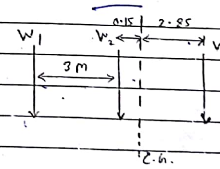
Sol



$$\bar{x} = \frac{\Sigma W_i \times x_i}{\Sigma W_i}$$

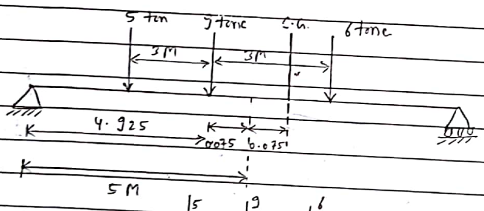
$$= \frac{5\times 0 + 9\times 3 + 6\times 6}{5+9+6}$$

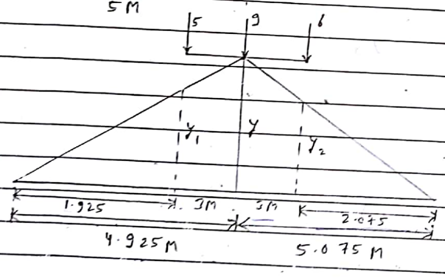
$$= 3.15 \text{ m}$$



Nearest load to C.G is 9 tone load which is also heavier then next, nearest load to C.G (6 tone). hence load under consideration is 9 tone load.

Imp 'Center of span will be mid way b/w. C.G. of load system & load under consideration' (9 tone load).



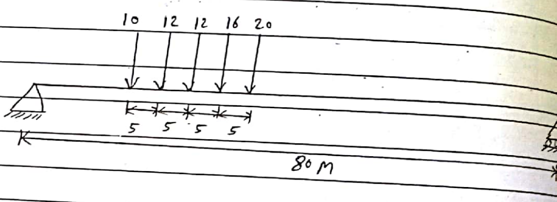


$y = \frac{a.b}{l} = \frac{4.925 \times 5.075}{10} = 2.5$

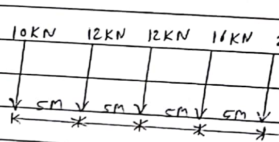
$y_1 = \frac{1.925}{4.925} \cdot y = 0.977$ $\quad y_2 = \frac{2.075}{5.075} \cdot y = 1.0$

$(B.M)_9 = 5 \times y_1 + 9 \times y + 6 \times y_2 = 33.54$

Q-2 5- point loads of 10, 12, 12, 16, 20 spaced at 5m c/c roll over a of span 80m & simply supported ends. Loads move left to with 20 KN load leading. Find absolute maximum B.M.

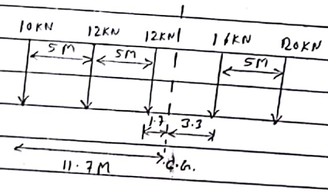


Sol



$\bar{x} = \frac{10 \times 0 + 12 \times 5 + 12 \times 10 + 16 \times 15 + 20 \times}{10 + 12 + 12 + 16 + 20}$

$= 11.7$



Nearest load to C.G. is 12 KN load & nearest load to C.G is 16KN which is heavier than nearest load to C.G. So

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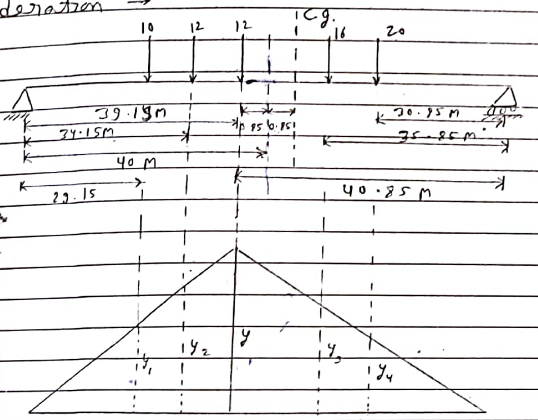
M_{max} may occur under any of these 2 loads check both the condition.

Assume nearest load to c.g as load under consideration →



ILD for $(BM)_{12}$

$$y = \frac{a.b}{l} = \frac{39.15 \times 40.85}{80} = 20\ M$$

$$y_1 = \frac{29.15}{39.15} \times y = 14.9 \qquad y_2 = \frac{34.15}{39.15} \times y = 17.45$$

$$y_3 = \frac{35.85}{40.85} \times y = 17.55 \qquad y_4 = \frac{30.85}{40.85} \times y = 15.1$$

$$(B.M)_{12} = 10 \times y_1 + 12 \times y_2 + 12 \times y + 16 \times y_3 + 20 \times y_4$$

$$= 1180\ KN\text{-}M$$

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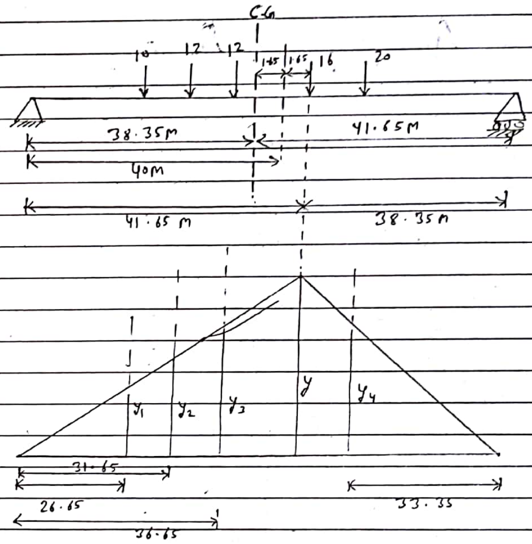

$$y = \frac{a.b}{l} = \frac{41.65 \times 38.35}{80} = \cancel{19.95}\ 19.915M$$

$$y_1 = \frac{19.915}{41.65} \times 26.65 = 12.77M \ ; \ y_2 = \frac{19.915}{41.65} \times 31.65 = 15.17$$

$$y_3 = \frac{19.915}{41.65} \times 36.65 = 17.57 \ ; \ y_4 = \frac{19.915}{38.35} \times 33.35 = 17.36$$

$$(BM)_{16} = 12.77 \times 10 + 15.17 \times 12 + 17.57 \times 12 + 19.915 \times 16 + 17.36 \times 20$$

$$= 1187.22\ KN\text{-}M$$

Absolute B.M for given case is 1187.22 KN-M

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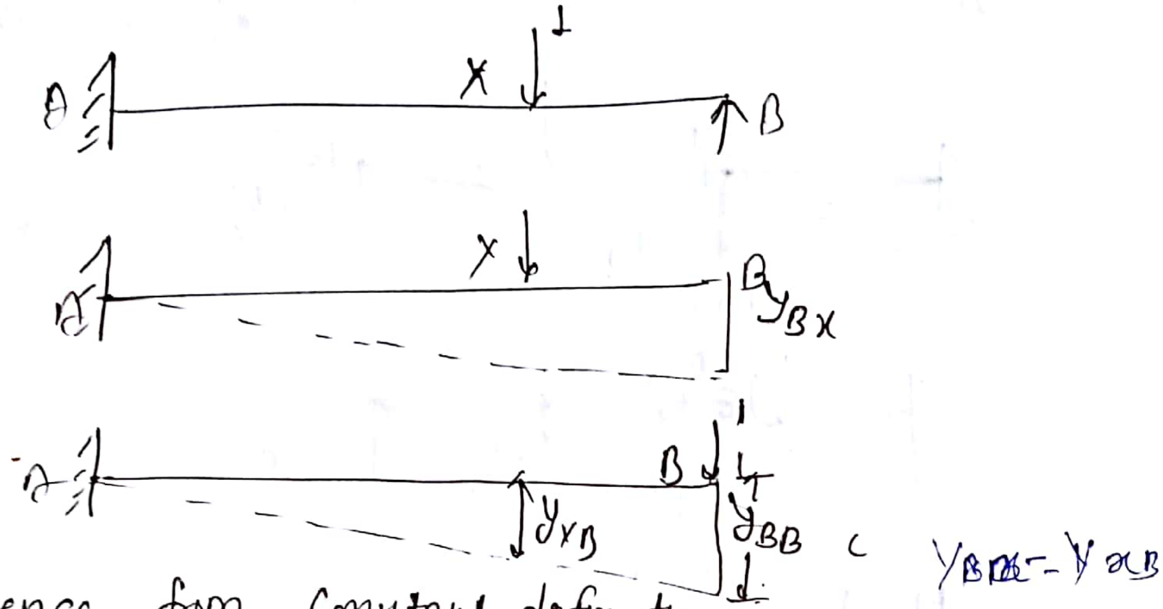
Propped Cantilever

(I) I.L for Prop reaction



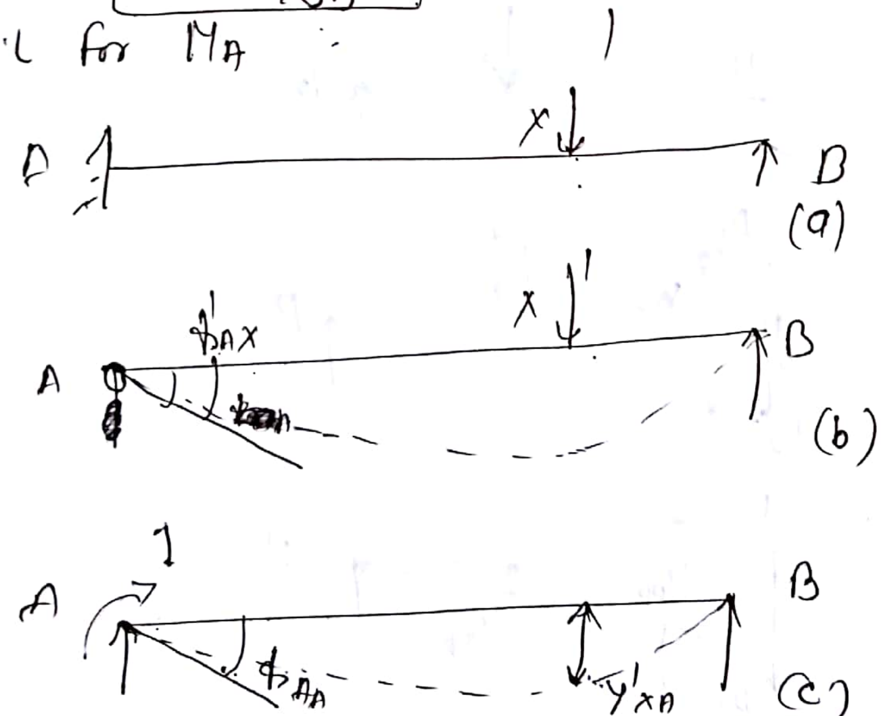
Hence from Consistent deformation

$R_B \cdot y_{BB} = y_{Bx}$

By Maxwell Reciprocal theorem

$$\boxed{R_B = \frac{y_{xB}}{y_{BB}}}$$

(II) I.L for M_A



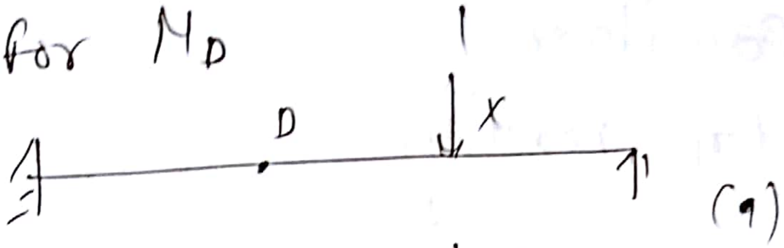
from Method of Consistent deformation

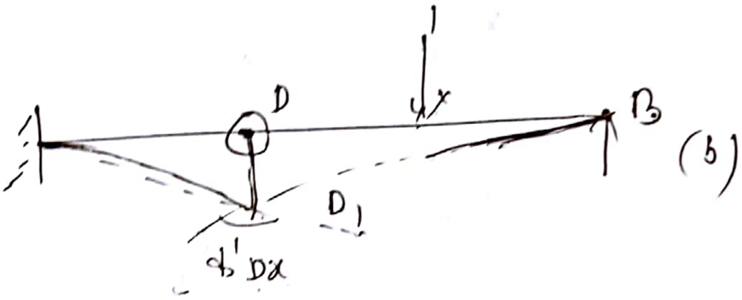
$M_A \cdot \phi_{AA} = \phi'_{Ax}$

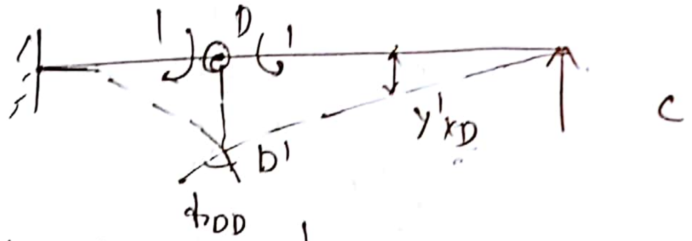
$\phi'_{Ax} = y'_{xA}$

$$\therefore \boxed{M_{A} = \frac{y'_{xA}}{\phi_{AA}}}$$

(3) I.L.D for M_D





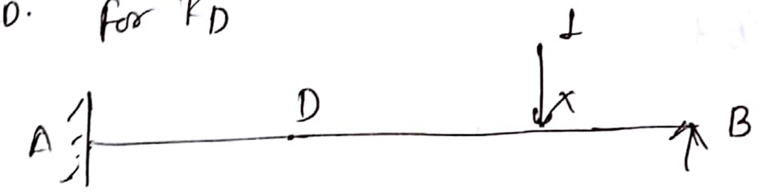


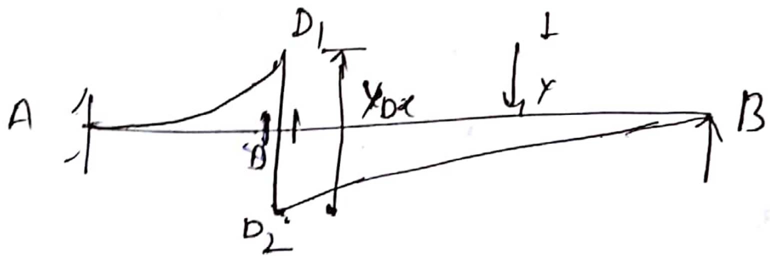
$M_D \cdot \phi_{DD} = \phi'_{Dx}$

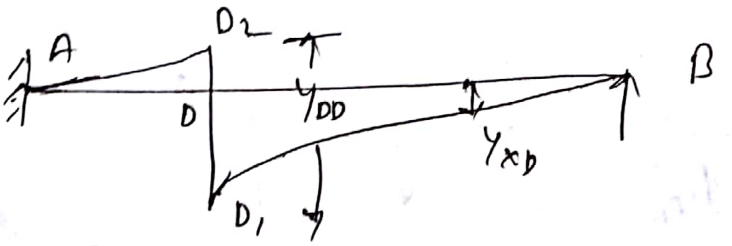
$\phi'_{Dx} = y'_{xD}$

$M_D \cdot \phi_{DD} = y'_{xD}$ or $M_D = \frac{y'_{xD}}{\phi_{DD}}$

(4) I.L.D. for P_D





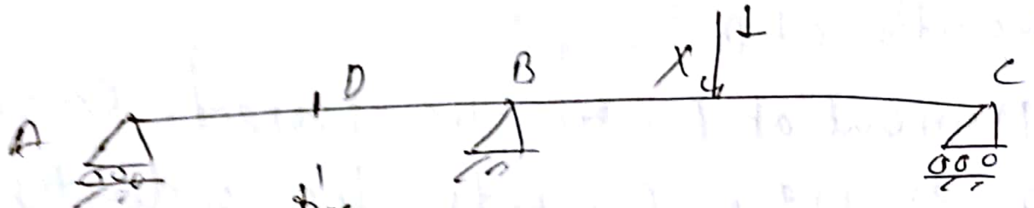


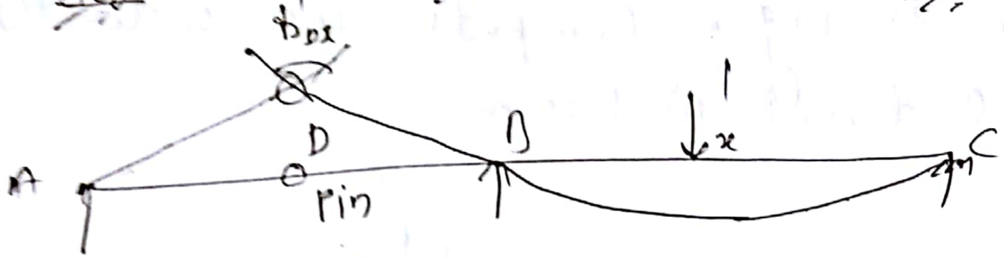
$P_D \cdot Y_{DD} = Y_{Dx}$

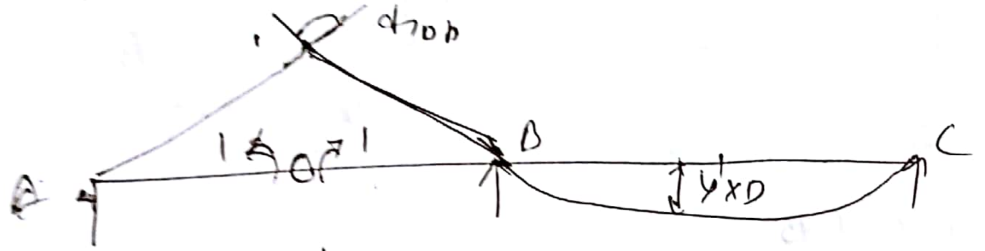
$Y_{Dx} = Y_{xD}$ (By Maxwell Reciprocal theorem)

$\boxed{P_D = \frac{Y_{xD}}{Y_{DD}}}$

Continious beam : Influence Line for Bending moments





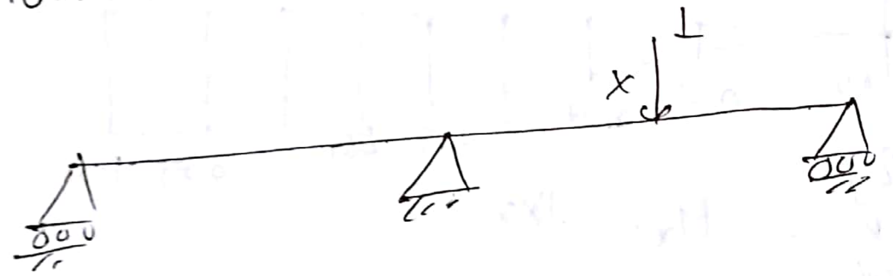


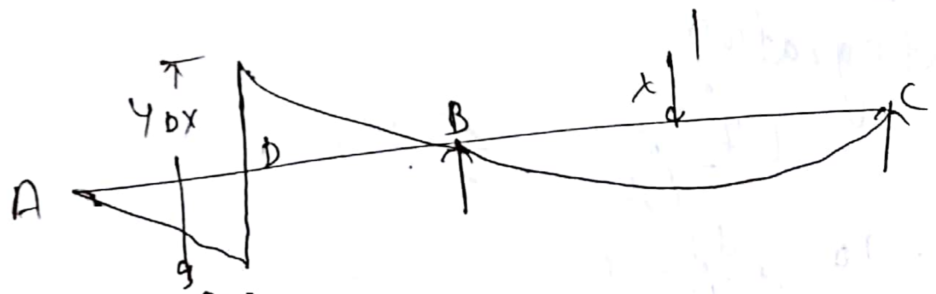
$$M_D = \frac{\phi'_{Dx}}{\phi_{DD}}$$

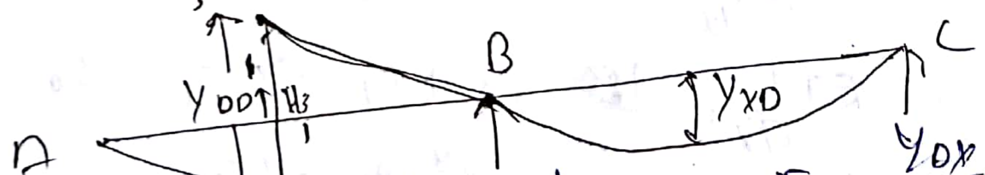
$\phi'_{Dx} = y'_{xD}$ (from Reciprocal theorem)

$$\boxed{M_D = \frac{y'_{xD}}{\phi_{DD}}}$$

Continious Beam: Influence line for S.F







$F_D \cdot Y_{DD} = Y_{Dx}$ or $F_D = \dfrac{Y_{Dx}}{Y_{DD}}$

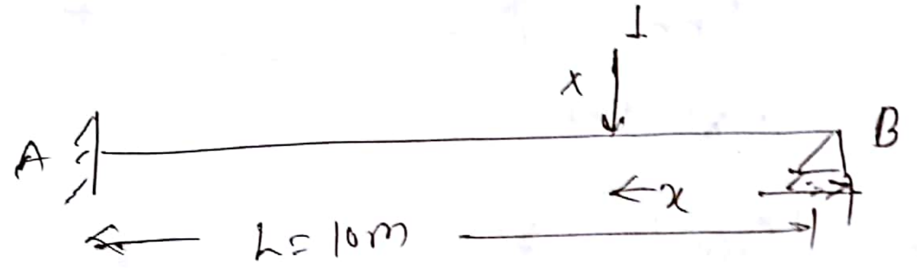
$\because Y_{Dx} = Y_{xD}$ (By Maxwell Reciprocal theorem) $F_D = \dfrac{Y_{xD}}{Y_{DD}}$

Q13 Draw the Influence lines for

(i) Reaction at B.

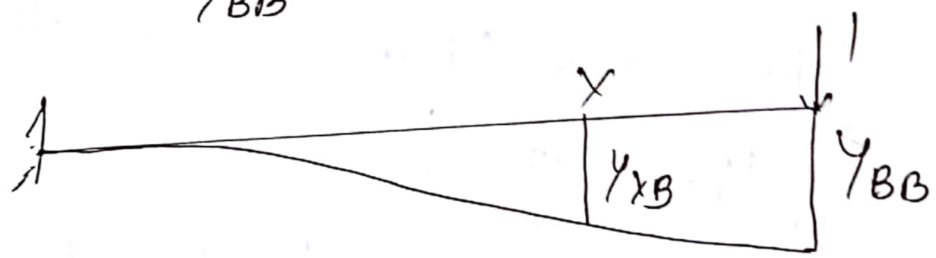
(ii) Moment at A for the Propped Cantilever shown in fig. Compute the ordinates at interval of 1.25 m.

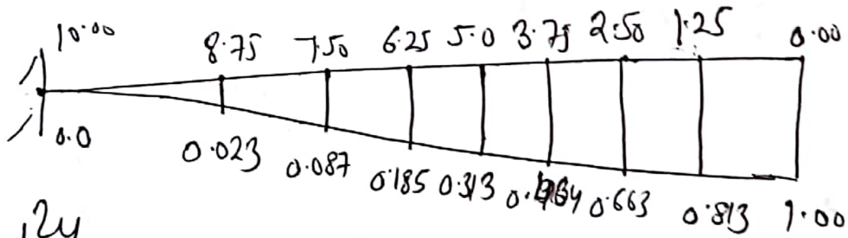
Sol[4]



(a) I.L. for R_B

$$R_B = \frac{Y_{xB}}{Y_{BB}}$$





$$EI\frac{d^2y}{dx^2} = -M_x = 1 \times x$$

Integrating

$$EI\frac{dy}{dx} = \frac{x^2}{2} + C_1$$

at $x = 10$, $\frac{dy}{dx} = 0$

$$EI\frac{dy}{dx} = 50 + C_1 \Rightarrow C_1 = -50$$

$$EI\,y = \frac{x^3}{6} + C_1x + C_2$$

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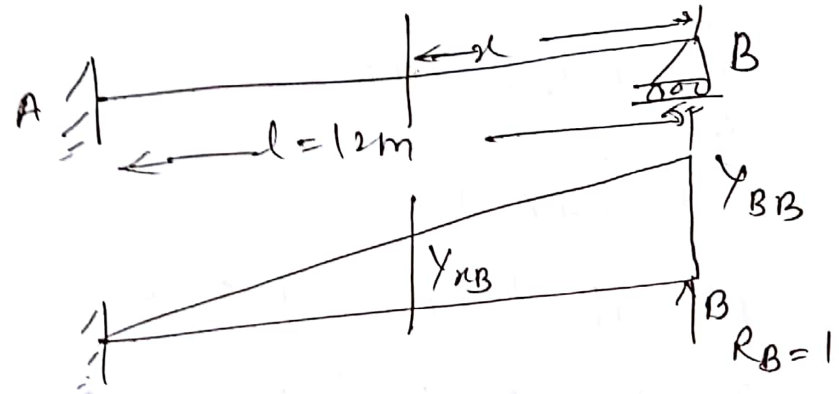
Q.1 Draw the IL for reaction at B and for the support M_A at A for the propped cantilever. Compute IL ordinates at 1.5 m intervals.

Sol^n



When $R_B = 1$. y_{xB} is the displacement at section x due to unit load applied at B

$$M_x = EI\frac{d^2y}{dx^2} = R_B \cdot x = -1 \cdot x$$

so $$EI\frac{d^2y}{dx^2} = -x$$

$$EI\frac{dy}{dx} = -\frac{x^2}{2} + C_1$$

$$EIy = -\frac{x^3}{6} + C_1x + C_2$$

At $x = 12$, $y = 0$, $\frac{dy}{dx} = 0$

$C_1 = 72$, $C_2 = -576$

$$y_{xB} = \frac{1}{EI}\left[-\frac{x^3}{6} + 72x - 576\right]$$

$$y_{BB}\ (\text{at } x=0) = \frac{-576}{EI}$$

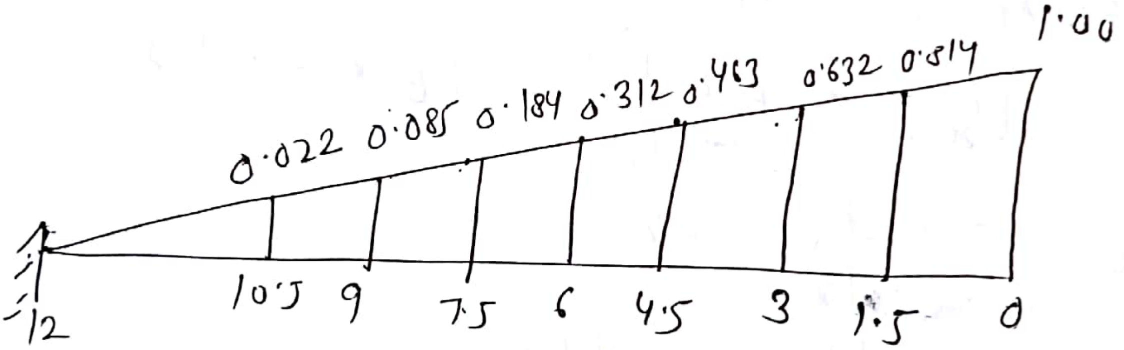
We know that ILD for R_B at x

$$R_B = \frac{y_{xB}}{y_{BB}} = \frac{\frac{1}{EI}\left[-\frac{x^3}{6}+72x-576\right]}{\frac{1}{EI}(-576)}$$

$$= \frac{-\frac{x^3}{6}+72x-576}{[-576]}$$

ILD for R_B at 1.5 m interval

x in m	0	1.5	3	4.5	6	7.5	9	10.5	12
R_B	1	0.814	0.632	0.463	0.312	0.184	0.085	0.022	0.00



(b)

To draw the IL for M_A. We have to introduce hinge at A and applied unit Rotation at A. we applied unit moment at A and find the general displacement at x from B.

Taking Moment about A

$$R_B \times 12 - 1 = 0 \qquad [M_A = 1]$$

$$R_B = \frac{1}{12}$$

Taking Moment about B

$$R_A \times 12 + 1 = 0$$

So $R_A = \frac{-1}{12}$

$$R_B = -R_A = \frac{1}{12}$$

$$M_x = -EI \frac{d^2y}{dx^2} = \frac{1}{12} \times x$$

$$EI \frac{dy}{dx} = -\frac{x^2}{24} + C_1$$

$$EI\, y = \frac{-x^3}{72} + C_1 x + C_2$$

At $x = 0$, $y = 0$

So $0 \cancel{+} = 0 + 0 + C_2$

$$C_2 = 0$$

At $x = 12$, $y = 0$

$$0 = -\frac{12 \times 12 \times 12}{72} + C_1 \times 12 + 0$$

$$C_1 = 2$$

$$y_{xA} = \frac{1}{EI}\left(-\frac{x^3}{72} + 2x\right)$$

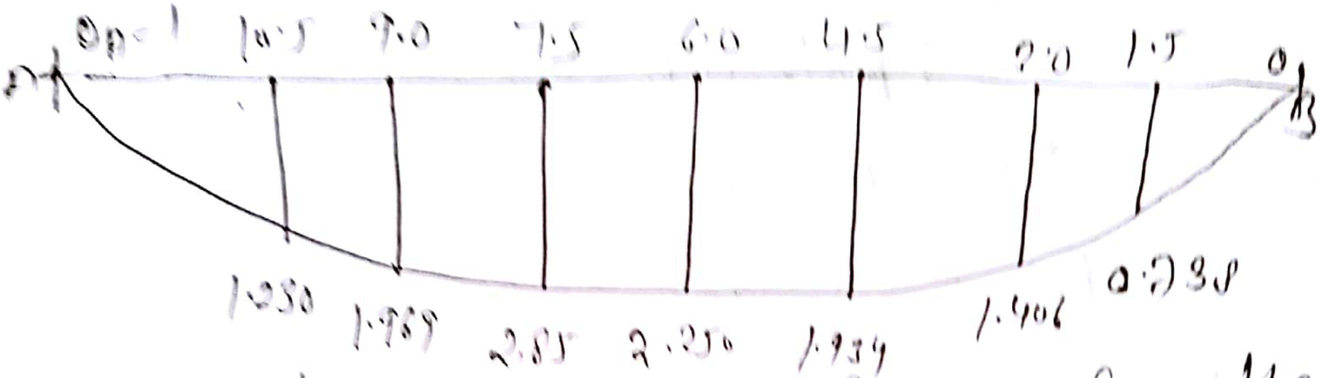
$$\frac{dy}{dx} = \theta_{xA} = \frac{1}{EI}\left(-\frac{x^2}{24} + 2\right)$$

$$\theta_{AA}\,(\text{at } x = 12) = \frac{-4}{EI}$$

$$M_A = \frac{Y_{xA}}{\theta_{AA}}$$

ILD for M_B

x	0	1.5	3.0	4.5	6.0	7.5	9.0	10.5	12.0
ILD for M_B	0.0	0.738	1.406	1.934	2.250	2.281	1.969	−1.250	0.0



Q. Draw the Influence line for R_A for the continuous beam. Determine IL ordinates at 1 m interval.

Sol^n If we have draw Influence line for R_A, Remove support A and apply unit force at A along R_A and Compute deflection at x on CB and BA.

Taking moment about C

$$R_A \times 10 + R_B \times 5 = 0$$

$$R_A \times 10 = -R_B \times 5$$

$$R_B = \frac{-R_A \times 10}{5} = \frac{-1 \times 10}{5} = -2$$

$$\Sigma V = 0$$

$$R_A + R_B + R_C = 0$$

$$R_C = 1$$

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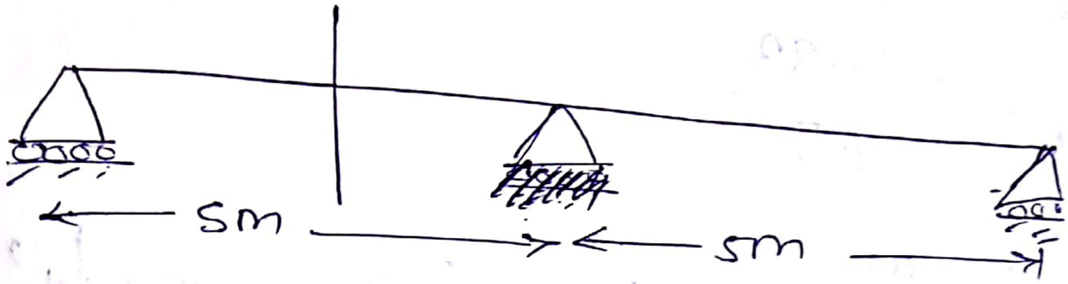
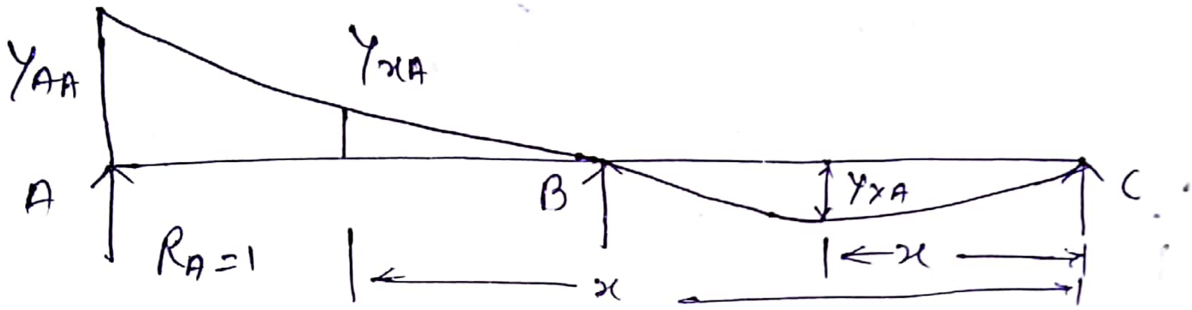




Now

$$M_x = \frac{-EI\,d^2y}{dx^2} = 1\cdot x \,\vdots - 2(x-5)$$

or

$$\frac{EI\,d^2y}{dx^2} = -x \,;\, +2(x-5)$$

$$\frac{EI\,dy}{dx} = \frac{-x^2}{2} + C_1 \,\vdots\, (x-5)^2$$

$$EI\,y = \frac{-x^3}{6} + C_1 x + C_2 \,;\, \frac{(x-5)^3}{3}$$

At $x = 0$, $y = 0$

At $x = 5$ $y = 0$

So $C_2 = 0$, $C_1 = 4.167$

$$Y_{xA} = \frac{1}{EI}\left[-\frac{x^2}{6} + 4.167x \,\vdots + \frac{(x-5)^3}{3}\right]$$

At $x = 10$

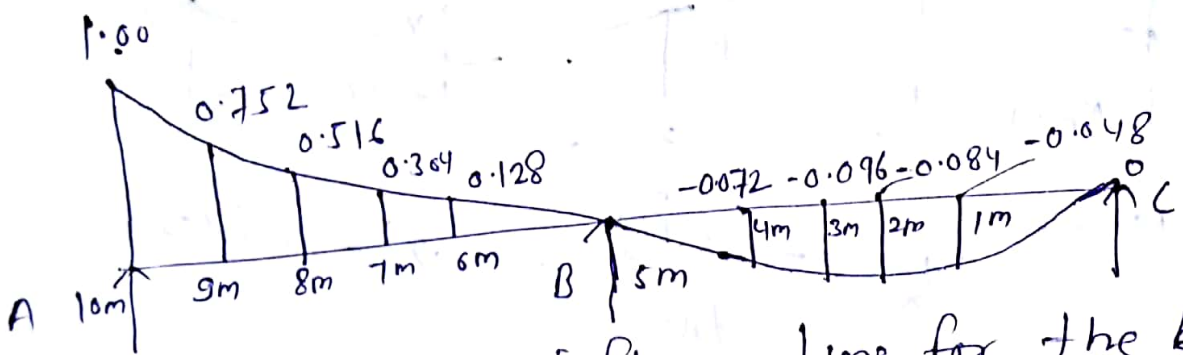
$$Y_{AA} = \frac{1}{EI}\left[-166.67 + 41.67 + 41.67\right]$$

ILD for M.

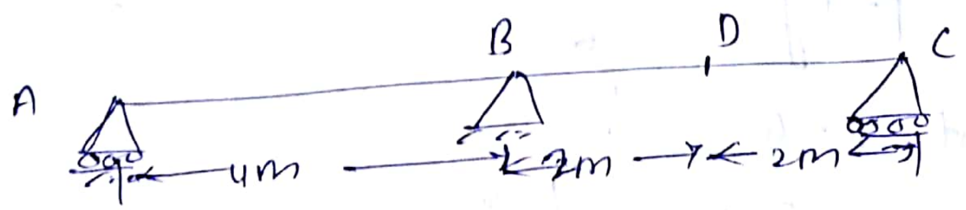
$$Y_{AA} = \frac{-83.33}{EI}$$

ILD at x

$$R_A = \frac{Y_{xA}}{Y_{AA}} = \frac{1}{83.33}\left[\frac{-x^3}{6} + 41.67x + \frac{(x-5)^3}{3}\right]$$



Q3 Determine the influence line for the bending moment at D, the middle point of span BC, of a continuous beam shown in fig. Compute the ordinates at 1m internal.



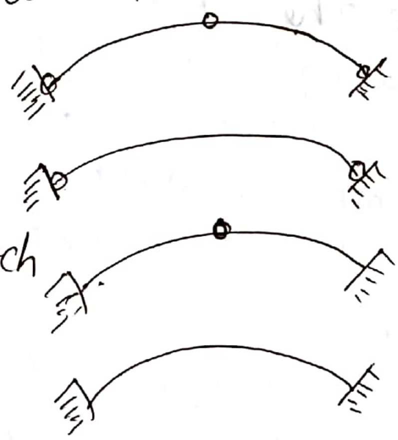
Unit II

Arch

An Arch is a curved girder supported at its ends and carrying transverse load which are frequently vertical. Since the transverse loading at any section normal to the axis of the girder is at an angle to the normal face, an arch is subjected to three restraining forces (i) thrust (ii) shear force (iii) bending moment.

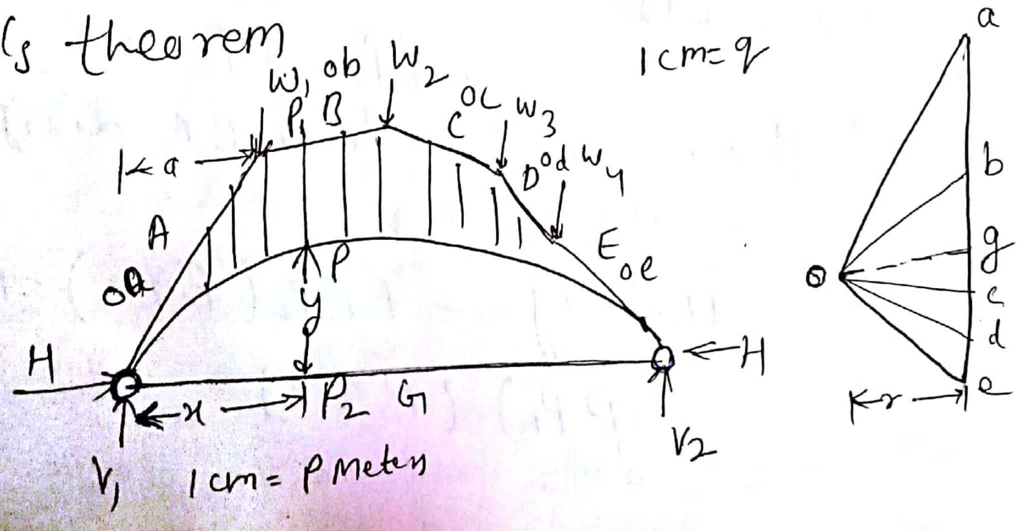
Depending upon the number of hinges, arches may be divided into four classes

(i) Three hinged arch
(ii) Two hinged arch
(iii) single hinged arch
(iv) fixed arch.



A Three hinged arch is statically determinate structure while the rest three arches are statically indeterminate.

Eddy's theorem



Theoretically the BM at P is given by

$$M_p = V_1 x - W_1(x-a) - Hy = \mu_x - Hy$$

where $\mu_x = V_1 x - W_1(x-a)$ = usual bending moment at a section due to load system on a simply supported beam.

Consider a section at P distant x from A on an arch. Let the other co-ordinate of P be y. For the given system of load the linear arch can be constructed. Since the funicular polygon represent the bending moment diagram to some scale the vertical intercept P_1P_2 at the section P will given the bending moment due to the external load system. If the arch is drawn to a scale of 1cm = P m load diagram is plotted to scale of 1cm = q N and if the distance of pole o from the load line is r, the scale of bending moments will be 1cm = p.q.r N-m

Then

$$\mu_x = (P_1P_2) \times \text{scale of B.M diagram}$$
$$= P_1P_2\ (p.q.r)$$
$$H.y = (PP_2) \times \text{Scale of B.M diagram}$$
$$= PP_2\ (p.q.r)$$
$$M_p = \mu_x - Hy = P_1P_2(p.q.r) - PP_2(p.q.r)$$
$$= (PP_1)(p.q.r)$$

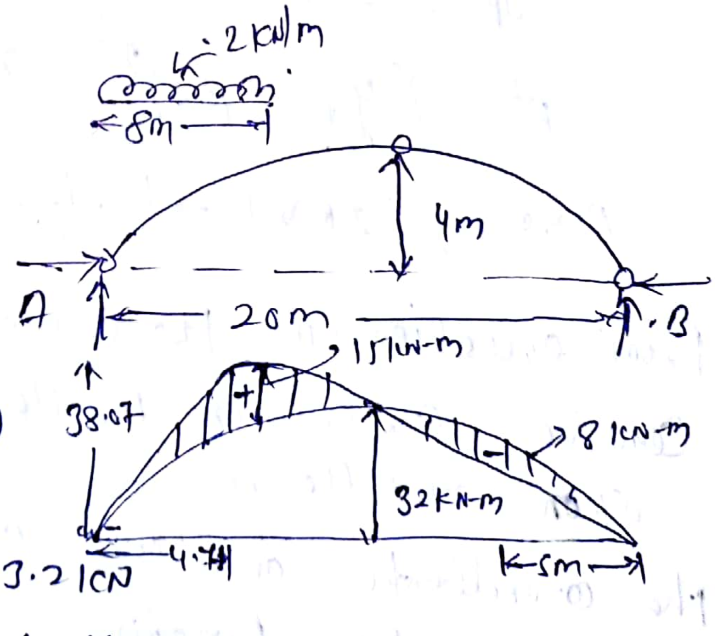
A parabolic arch hinged at the springings and crown has a span of 20 m. The central rise of the arch is 4 m. It is loaded with udl of intensity 2 kN/m on the left 3 m length. Calculate

(a) The direction and magnitude of reaction at the hinges

(b) The bending moment, normal thrust and radial shear at 4 m and 15 m from the left end. (c) Maximum positive and negative bending moments.

Sol^n

(a) Reaction at the hinges



For vertical Rx^n take moment about B

$V_A \times 20 = 2 \times 8\,(20-4)$

$V_A = 12.8$ kN

$V_B = 8 \times 2 - 12.8 = 3.2$ kN

Since the B.M at $M_c = 0$

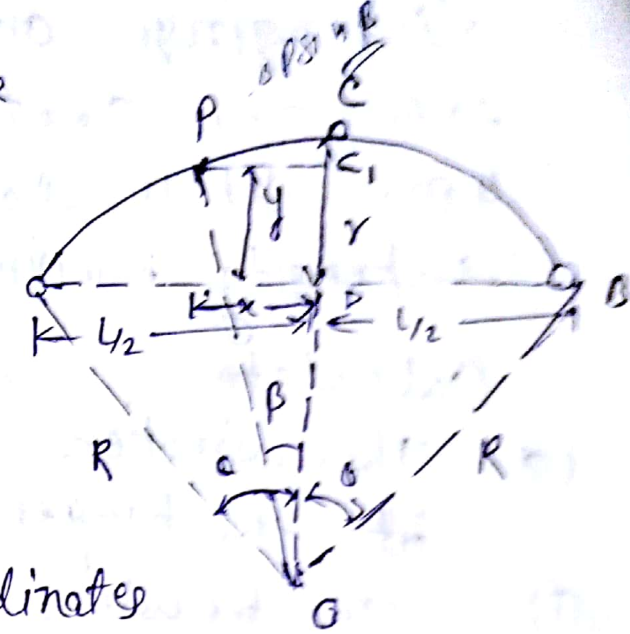
$M_c = (3.2 \times 10) - H \times 4 = 0$

$H = \frac{32}{4} = 8$ kN

∴ Reaction at A

$R_A = \sqrt{V_A^2 + H^2} = \sqrt{(12.8)^2 + (8)^2} = 15.09$ kN

Three hinged circular Arch



Let us now consider the Centre line of the arch to be segment of circle of radius R, subtending on angle of 2θ at the centre.

Let (x,y) be the Co-ordinates of the point P. Draw line Parallel to AB

Then

$$OP^2 = OC_1^2 + PC_1^2$$

$$R^2 = \{y + (R-r)\}^2 + x^2 \quad \text{— (I)}$$

Also $r(2R-r) = \frac{L}{2} \cdot \frac{L}{2} = \frac{L^2}{4}$ — (II) (Properties of circle)

from equation (ii) the value of the radius Can be Calculated for the known values of span and the rise

The co-ordinate of $P(x,y)$ can also be expressed as trigonometric function

$$x = OP \sin\beta = R\sin\beta$$

$$y = C_1D = OC_1 - OD = R\cos\beta - R\cos\theta$$

$$= R(\cos\beta - \cos\theta)$$

$$= R\cos\beta - R\cos\theta$$

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$R_B = \sqrt{V_B^2 + H^2} = \sqrt{(3.2)^2 + 8^2} = 8.62\, kN$

Its inclination with the horizontal

$$\tan\theta_A = \frac{V_A}{H} = \frac{12.8}{8} = 1.6$$

$$\theta_A = 58^\circ$$

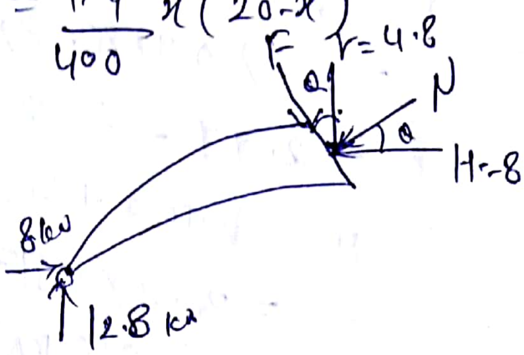
$$\tan\theta_B = \frac{V_B}{H} = \frac{3.2}{8} = 0.4 \Rightarrow \theta_B = 21^\circ.48'$$

(b) B.M, thrust & Radial shear

$$y = \frac{4r}{L^2} x(L-x) = \frac{4\times4}{400} x(20-x)$$

$$= \frac{x}{25}(20-x)$$

$$\frac{dy}{dx} = \frac{20-2x}{25}$$



At $x=4$ Then

$$y = \frac{4}{25}(20-4) = 2.56\, m$$

$$\tan\theta = \frac{dy}{dx} = \frac{20-2\times4}{25} = 0.48$$

$$\theta = 25^\circ 38'$$

$\sin\theta = 0.433$ and $\cos\theta = 0.901$

$$M_4 = +(12.8\times4) - (4\times2\times2) - (8\times2.56)$$

$$= 14.72\ kN\text{-}m$$

vertical shear at the section

$V = 12.8 - 2\times4 = 4.8\, kN$ and $H = 8\, kN$

Then Radial shear

$$F = V\cos\theta - H\sin\theta = 4.8\times0.901 - 8\times0.433$$

$$= +0.861\, kN$$

$F = 0.861\, kN\ (\uparrow\downarrow)$

$$N = V\sin\theta + H\cos\theta = 4.8\times4.33 + 8\times0.901$$

$$= 9.286\, kN$$

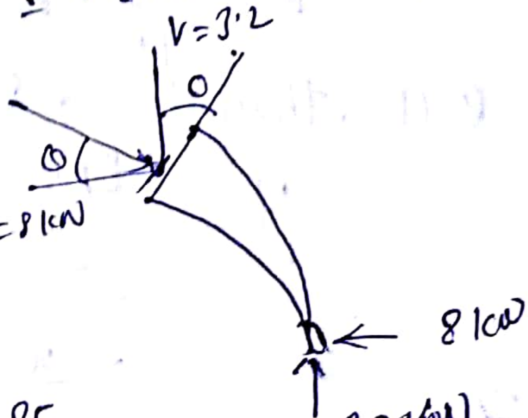
At $x = 15$

$$y = \frac{15}{25}(20 - 15) = 3.0\,m$$

$$\frac{dy}{dx} = \tan\theta = \frac{20 - 2\times 15}{15} = -0.4$$

$\therefore \theta = 21°.48'$, $\sin\theta = 0.3714$ and $\cos\theta = 0.9285$

$$M_{15} = (+3.2 \times 5) - 8(3.0) = -8\ kN\text{-}m$$



$F = V\cos\theta - H\sin\theta$

$= 3.2 \times 0.9285 - 8 \times 0.3714$

$= 2.97 - 2.97 = 0$

$N = V\sin\theta + H\cos\theta$

$= 3.2 \times 0.3174 + 8 \times 0.9285$

$= 8.616\ kN$

Maximum positive and negative B.M

Maximum positive B.M will occur somewhere under

Maximum positive B.M will occur at x from the left the wall. Let it occur

hinge

$$M_x = +(12.8 \times x) - \frac{2x^2}{2} - 8y = 12.8x - x^2 - \frac{8x}{25}(20 - x)$$

$$\frac{dM_x}{dx} = 0 = +12.8 - 2x - \frac{32}{5} + \frac{16}{25}x = 0$$

for them $x = 4.7\,m$

$$M_{ax}(+ve) = 12.8 \times 4.7 - 4.7^2 - \frac{8}{25}(4.7)(20 - 4.7)$$

$= 15\ kN$

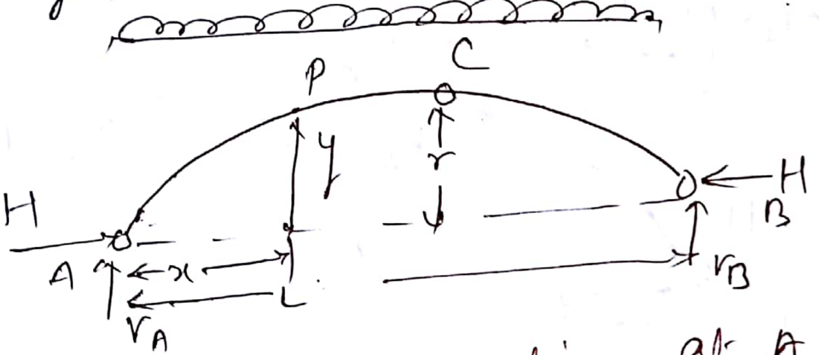
Maximum Negative B.M will occur somewhere in the portion BC for which the equation of B.M

$$M_x = +3.2x - 8y = +3.2x - \frac{8x}{25}(20 - x)$$

$$\therefore \frac{dM_x}{dx} = 0 = +3.2 - \frac{32}{5} + \frac{16x}{25}$$ from which $x = 5\,m$

$$M_{max(-ve)} = 3.2 \times 5 - \frac{8 \times 5}{25}(20 - 5) = -8\ kN\text{-}m$$

A symmetrical parabolic Arch with a Central hinge of rise r and span L is supported at its ends on pin at the same level. What is the value of the horizontal thrust when a load W which is uniformly distributed horizontally covers the whole span, show also that with this loading there is no bending moment at any point in the arch rib.



Sol^n The vertical reaction at A & B will be equal to $\frac{W}{2}$

for value of H taking moment about C.

$$\frac{W}{2}\times\frac{L}{2}-\frac{W}{2}\times\frac{L}{4}-H\times r=0$$

$$\Rightarrow H=\frac{WL}{8r}$$

Let us now consider any section at distance x from A.

equation of parabola $y=\frac{4r}{L^2}x(L-x)$

$$M_x=-H\cdot y+V_A\times x-\frac{W}{L}\frac{x^2}{2}=-\frac{WL}{8r}\cdot\frac{4r}{L^2}x(L-x)+\frac{W}{2}\cdot x-\frac{Wx^2}{2L}$$

$$=-\frac{Wx}{2}+\frac{Wx^2}{2L}+\frac{Wx}{2}-\frac{Wx^2}{2L}=0$$

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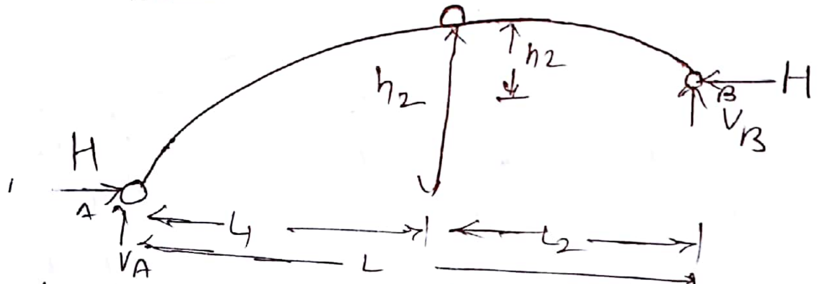
Q3 An Arch in the form of a Parabola with axis vertical has hinges at the abbutments and vertex. The abutments are at different levels, the horizontal span being L and the height of vertex above the abutment being h_1 & h_2.

Show that horizontal thrust due to a load ω/unit length udL across the span is

$$\frac{\omega L^2}{2(\sqrt{h_1}+\sqrt{h_2})^2}$$



Sol^n Let L_1 be the distance of vertex from the left hand abutment. with C as origin. the eq^n of parabola

$$y = Kx^2$$

For CA therefore $h_1 = KL_1^2 \Rightarrow K = \frac{h_1}{L_1^2}$ -(1)

For CB $h_2 = K(L-L_2)^2$

$$K = \frac{h_2}{(L-L_1)^2} \quad (11)$$

Equating the two equations

$$\frac{h_1}{L_1^2} = \frac{h_2}{(L-L_1)^2}$$

$$L_1^2 h_2 = (L-L_1)^2 h_1$$

or $L_1\sqrt{h_2} = (L-L_1)\sqrt{h_1}$

$$L_1 = \frac{L\sqrt{h_1}}{\sqrt{h_1}+\sqrt{h_2}}$$

By taking

Taking Moment about BC

$$H h_1 = V_A \frac{L\sqrt{h_1}}{\sqrt{h_1}+\sqrt{h_2}} - \frac{w}{2}\frac{h_1 L^2}{(\sqrt{h_1}+\sqrt{h_2})^2} \quad \text{—(iii)}$$

By taking Moment about B

$$H(h_1-h_2) + \frac{wL^2}{2} = V_A L$$

$$\text{or } V_A = \frac{H(h_1-h_2)}{L} + \frac{wL}{2} \quad \text{—(iv)}$$

Put the value of V_A in eq (iii)

$$H h_1 = \left[\frac{H(h_1-h_2)}{L} + \frac{wL}{2}\right]\frac{L\sqrt{h_1}}{\sqrt{h_1}+\sqrt{h_2}} - \frac{w}{2}\frac{h_1 L^2}{(\sqrt{h_1}+\sqrt{h_2})^2}$$

$$\Rightarrow H\left[h_1 - \frac{(h_1-h_2)\sqrt{h_1}}{\sqrt{h_1}+\sqrt{h_2}}\right] = \frac{wL^2\sqrt{h_1}}{2(\sqrt{h_1}+\sqrt{h_2})} - \frac{w h_1 L^2}{2(\sqrt{h_1}+\sqrt{h_2})^2}$$

$$H\cdot\sqrt{h_1 h_2} = \frac{wL^2}{2(\sqrt{h_1}+\sqrt{h_2})^2}\sqrt{h_1 h_2}$$

$$H = \frac{wL^2}{2(\sqrt{h_1}+\sqrt{h_2})^2}$$

Q.4 A Three hinged parabolic arch of 20 m ~~metre~~ and 4m central rise carries a point load of 4 KN at 4m horizontally from the left hand hinge. calculate the normal thrust and radial shear at the section under the load. Also calculate the Maximum B.M positive and negative.

mn

Sol^n when taking origin A then eq^n of Parabola

$$y = \frac{4r}{L^2}x(L-x) = \frac{4\times4}{20\times20}x(20-x) = \frac{x}{25}(20-x)$$

For V_A taking Moment about B, we get

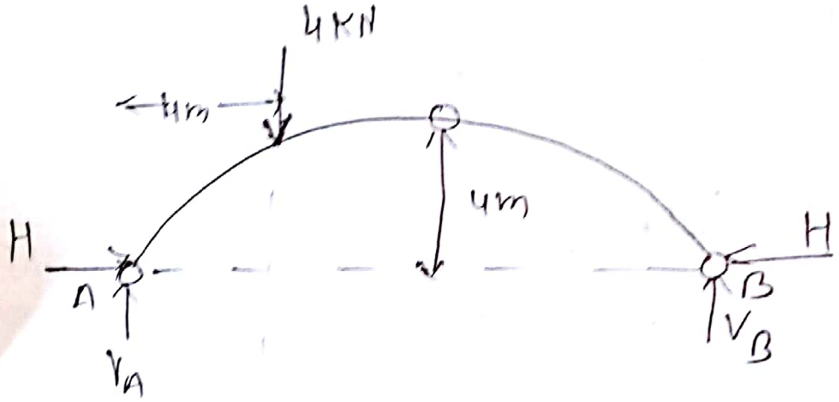
$$20V_A + 4\times16 = 0$$

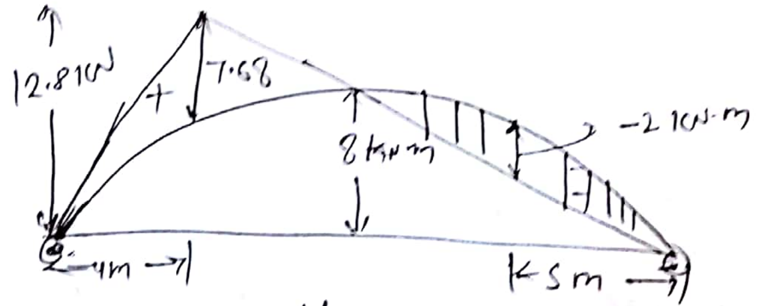
or $V_A = \frac{64}{20} = 3.2\ kN\ (\uparrow)$

$V_B = 4 - 3.2 = 0.8\ kN\ (\uparrow)$

For value of H

$$M_C = 4H - (0.8\times10) = 0 \Rightarrow H = \frac{8}{4} = 2\ kN$$





B.M at any section

$$M_x = \mu_x - Hy$$

The μ_x diagram is a triangle having maximum ordinate $= 3.2\times4 = 12.8$ kN m under the point load.

The Hy diagram is a parabola having a maximum ordinate $= 2\times4 = 8$ kN m under central hinge

at $x = 4$ $\quad y = \frac{4}{25}(20-4) = 2.56$

$$M_p = (3.2\times4) - (2\times2.56) = 7.68\ kN\,m$$

Maximum negative B.M will occur somewhere in the portion BC. ~~Her~~

$$M_x = 0.8x - 2y = +0.8x - 2 \cdot \frac{x}{25}(20-x)$$

$$\frac{dM_x}{dx} = +0.8 - \frac{40}{25} + \frac{4}{25}x = 0 \text{ ~~which~~}.$$

$\Rightarrow x = 5\,m$

$M_{Max(-ve)} = (0.8 \times 5) - \frac{2}{25}\cancel{5}(20-5) = -2\ kN\text{-}m$

The equation of parabola

$$y = \frac{x}{25}(20-x)$$

$$\tan\theta = \frac{dy}{dx} = \frac{20}{25} - \frac{2x}{25}$$

$\therefore \tan\theta = (x=4) = 0.8 - 0.32 = 0.48$

$\therefore \theta = 25°38'$, $\sin\theta = 0.433$; $\cos\theta = 0.901$

Considering the point load slightly to the right of P. weget

$F = -H\sin\theta + V_A\cos\theta =$

$= -(2 \times 0.433) + (3.2 \times 0.901) = +2.017\ (\uparrow\downarrow)$

$N = H\cos\theta + V_A\sin\theta$

$= 2 \times 0.901 + 3.2 \times 0.433 = 3.188\ kN$

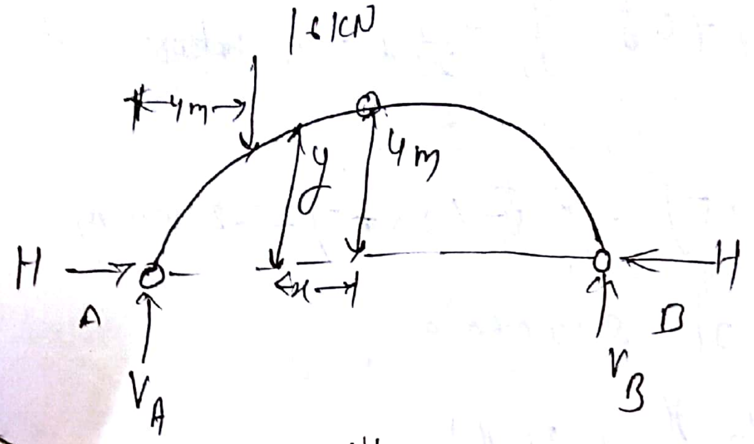
Q5 A symmetrical three hinged circular arch has a span of 16 m and a rise of the central hinge of 4m. It carries a vertical load of 16KN at 4m from the left hand end,

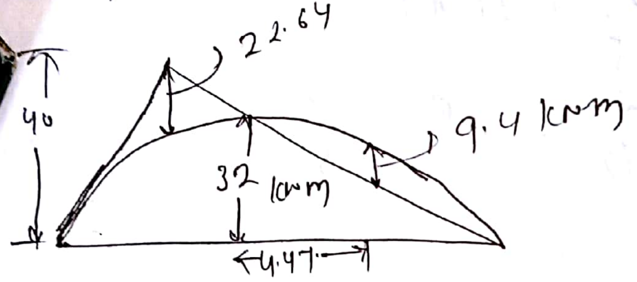
(a) find the magnitude of the thrust at the springings

(b) The reaction at the support

(c) bending moment at 6 m from the left hinge.

(d) the maximum positive and negative moment.





By properties of circle,

$4(2R-4) = 8 \times 8$

$\Rightarrow R = 10\,m$

Let y be the rise at any point at a distance x from the centre

Then $R^2 = x^2 + \{(R-r)^2 + y\}^2$

$10^2 = x^2 + \{(10-4) + y\}^2$

$\Rightarrow (6+y)^2 = 100 - x^2$

$y = (100 - x^2)^{1/2} - 6$

For vertical reaction

$V_A = \dfrac{16 \times 12}{16} = 12\,kN$

$V_B = 4\,kN$

Moment

Taking Moment about C

$$H \times 4 = (12 \times 8) - 16 \times 4 = 32$$

$$H = 8 \text{ KN}$$

(b) Reaction at A

$$R_A = \sqrt{V_A^2 + H^2} = \sqrt{144 + 64} = 14.42 \text{ KN}$$

$$\tan\theta_A = \frac{12}{8} = 1.5 \therefore \theta = 56°18'$$

$$R_B = \sqrt{V_B^2 + H^2} = \sqrt{16 + 64} = 8.94 \text{ KN}$$

$$\tan\theta_B = \frac{4}{8} = 0.5 \therefore \theta = 26°34'$$

(c) At 6 M from the left hinge

$$x = (8-6) = 2 \text{ M}$$

$$y = (100 - 2^2)^{1/2} - 6 = 9.8 - 6 = 3.8 \text{ M}$$

$$M = (12 \times 6) - (8 \times 3.8) - (16 \times 2) = +9.6 \text{ KN-M}$$

(d) The Maximum positive Moment will occur under the load; $x = (8-4) = 4 \text{ M}$

$$y = (100 - 4^2)^{1/2} - 6 = 3.17 \text{ M}$$

$$M_{max}(+ve) = (12 \times 4) - (8 \times 3.17) = +22.64 \text{ KN-M}$$

Max negative B.M will occur somewhere in CB. it occurs at a distance x from C, on the right hand side

$$y = (100 - x^2)^{1/2} - 6$$

$$M_x = +4(8-x) - 8\{(100-x^2)^{1/2} - 6\}$$

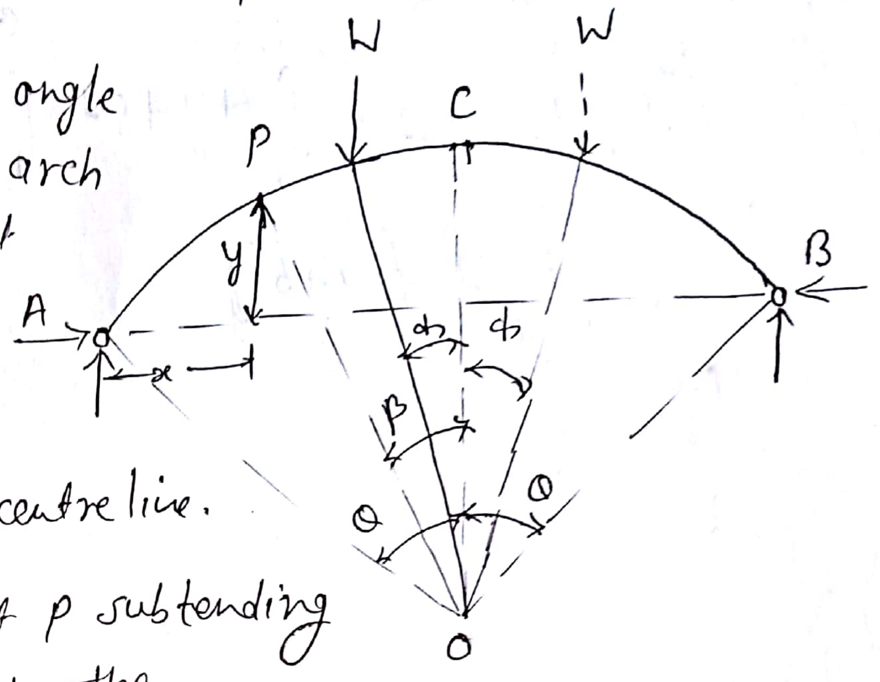
$$\therefore \frac{dM_x}{dx} = -4 - \frac{8(-2x)}{2(100-x^2)^{1/2}} = 0 \quad \text{or}$$

$$(100-x^2)^{1/2} = 2x \quad \text{or} \quad 5x^2 = 100 \Rightarrow x = \sqrt{20} = 4.47 \text{ M}$$

$$y = (100 - 4.47^2)^{1/2} - 6 = 2.94 \text{ M}$$

$$\therefore M_{max}(-ve) = -8 \times 2.94 + 4(8 - 4.47) = -23.82 + 14.12 = -9.4 \text{ KN-M}$$

Hinged circular Arch : Expression for H



θ be the half angle subtended by the arch at the Centre. Let the load W be acting at a section which makes an angle φ with the centre line.

Consider any point P subtending an angle β with the centre line

The Co-ordinates of P are given by

$x = R(\sin\theta - \sin\beta)$ - (1)

$y = R(\cos\beta - \cos\theta)$ - (ii)

Now $ds = R\,d\beta$

Now $\int_A^B y^2 ds = 2\int_0^\theta R^2(\cos\beta - \cos\theta)^2 R\,d\beta$

$= 2R^3 \int_0^\theta (\cos^2\beta - 2\cos\beta\cos\theta + \cos^2\theta)\,d\beta$

$= 2R^3 \left[\int_0^\theta \cos^2\beta\,d\beta - 2\cos\theta\int_0^\theta \cos\beta\,d\beta + \cos^2\theta\int_0^\theta d\beta\right]$

$\int_A^B y^2 ds = \frac{R^3}{2}(4\theta\cos^2\theta + 2\theta - 3\sin 2\theta)$

$= R^3(2\theta + \theta\cos 2\theta - 1.5\sin 2\theta)$

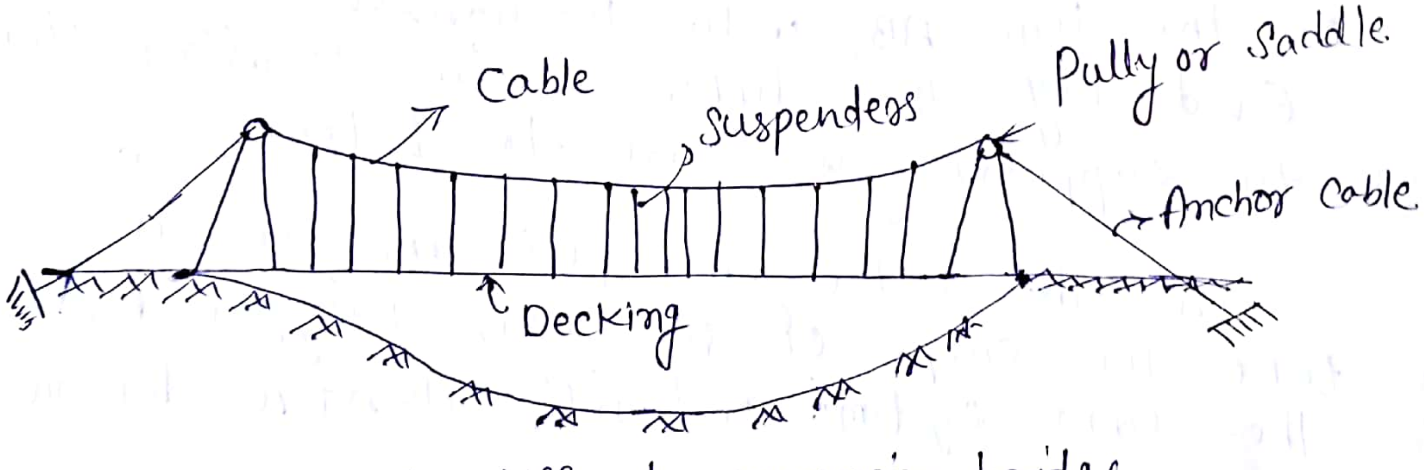
To find $\int \mu y\,ds$, assume an equal load W placed symmetrically on the other side so that the Configuration. V_A

Cable and suspension Bridges (Unit III)
By (Ganesh Sharma)

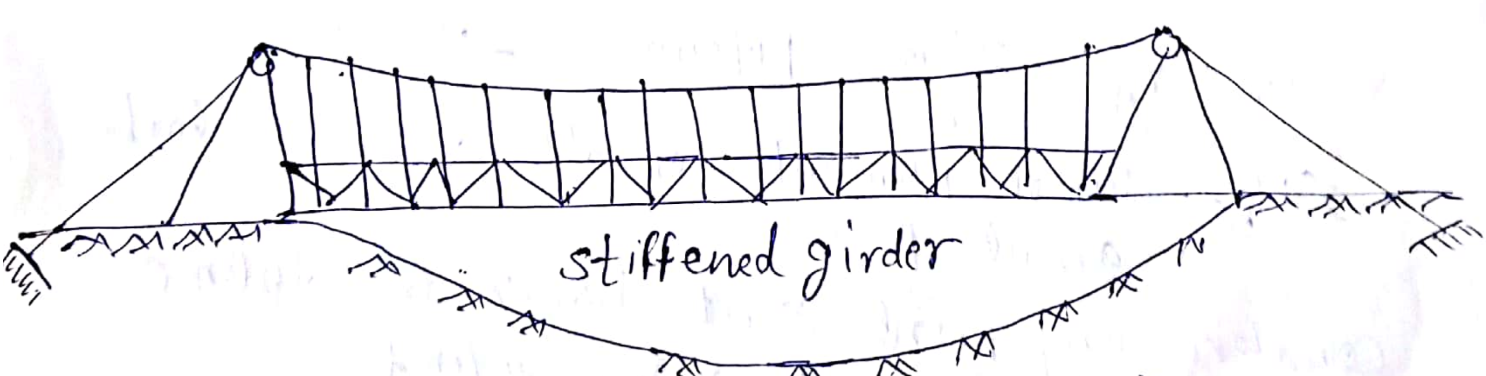
Introduction ⇒

Suspension bridges are used for highways where the span is more than 200 m. A suspension bridge consists of the following elements

(I) The cable
(II) Suspenders
(III) decking, including the stiffening girder
(IV) supporting tower
(V) anchorage

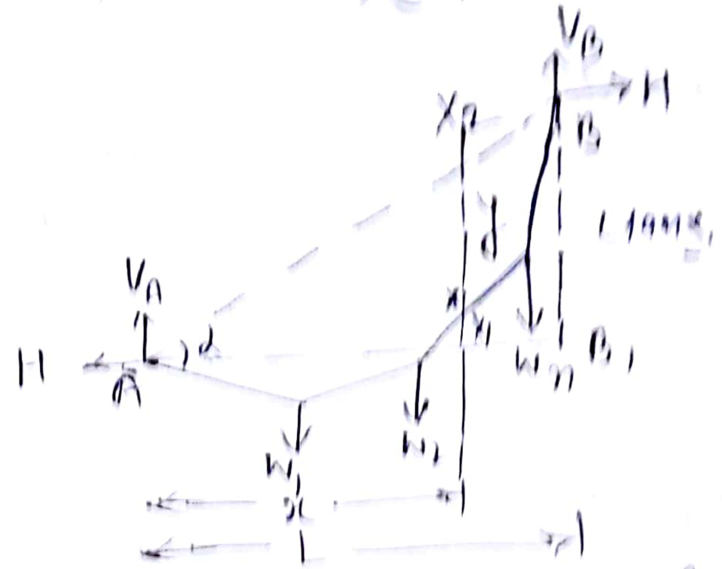


Unstiffened suspension bridge



stiffened suspension bridge

General Cable Theorem ⇒ (Equilibrium of light, c/)



A light cable suspended from two points A and B and subjected to a number of point load $W_1, W_2 \ldots W_n$. Let L be the horizontal span of the cable and α be the inclination of the line AB, with the horizontal.
Evidently, the difference in elevation b/w two supports is equal to $L\tan\alpha$.

Since the cable is in equilibrium it will take the shape of funicular polygon for the load system and will therefore deform as shown

For vertical reaction V_A. Taking moment about B

$$+V_A \times L + H \cdot L\tan\alpha - \Sigma M_B = 0$$

$$\text{or } V_A = \frac{\Sigma M_B}{L} - H\tan\alpha \quad -(1)$$

ΣM_B = Sum of moments of all external loads about B.

Consider any point x at horizontal distance x from A. so that $X_1 X_2 = x\tan\alpha$

Assuming that cable is perfectly flexible so that bending moment at any point on the cable is zero.

so that

$$+H(XX_1) + V_A \cdot x \pm \Sigma M_x = 0$$

$$+H(x\tan\alpha - y) + V_A \cdot x \pm \Sigma M_x = 0 \quad — (2)$$

M_x = sum of moments of all forces to the left of X

$y = XX_2$

put value of V_A in eq^n (2)

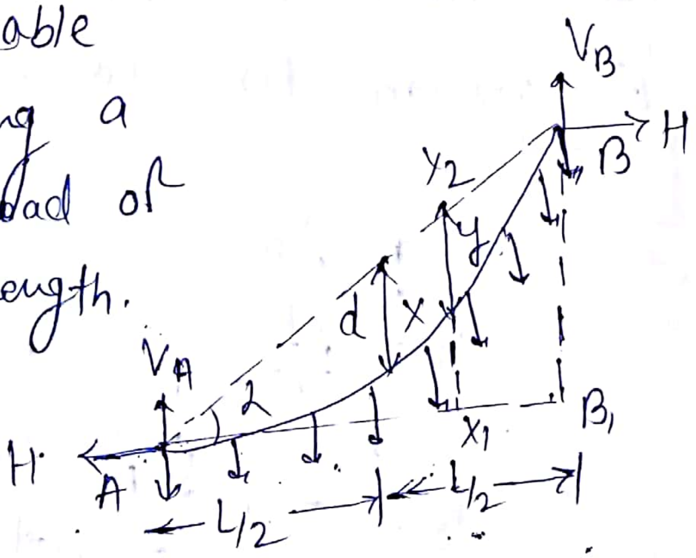
$$+H(x\tan\alpha - y) + \left\{ \frac{\Sigma M_B}{L} - H\tan\alpha \right\} x \pm \Sigma M_x = 0$$

or $$Hy - \frac{x}{L}\Sigma M_B + \Sigma M_x = 0$$

or $$\boxed{Hy = \frac{x}{L}\Sigma M_B - \Sigma M_x}$$ is the General cable theorem.

Uniformly loaded cable

In fig cable supporting a uniformly distributed load of intensity p per unit length.



By General Cable theorem

$$Hy = \frac{x}{L}\Sigma M_B - \Sigma M_x$$

$y = XX_2$

$$\Sigma M_B = P \cdot L \cdot \frac{L}{2} = P \cdot \frac{L^2}{2}$$

$$\Sigma M_x = P x \cdot \frac{x}{2} = P \frac{x^2}{2}$$

$$\because Hy = \frac{x}{L} \cdot P \cdot \frac{L^2}{2} - P \cdot \frac{x^2}{2}$$

$$= P \frac{L \cdot x}{2} - P \cdot \frac{x^2}{2} \quad - (1)$$

At the mid span $x = \frac{L}{2}$, $y = d$ = dip of cable

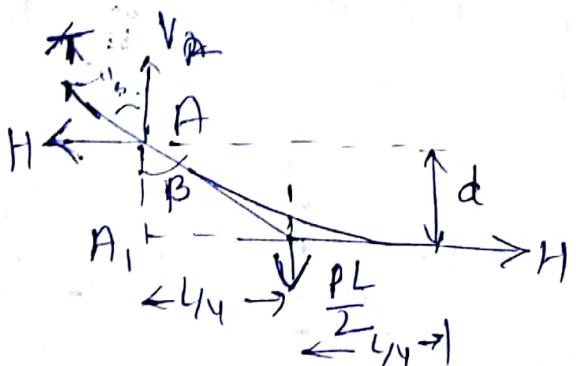
$$\therefore Hd = P \cdot \frac{L}{2} \cdot \frac{L}{2} - \frac{P}{2}\left(\frac{L}{2}\right)^2$$

$$= P \frac{L^2}{8}$$

$$\Rightarrow \boxed{H = P \cdot \frac{L^2}{8d}}$$

It gives the expression for the horizontal reaction H and is valid whether the cable chord is inclined or horizontal.

(b) Expression for Cable Tension at the Ends



The Cable Tension T at any resultant of vertical and horizontal reaction at the end they

$$T_A = \sqrt{V_A^2 + H^2} \quad \text{and} \quad T_B = \sqrt{V_B^2 + H^2}.$$

from

from fig

$$V_A = V_B = \frac{PL}{2}$$ (When Cable Chord is horizontal)

$$T_A = T_B = T = \sqrt{\left(\frac{PL}{2}\right)^2 + \left(\frac{PL^2}{8d}\right)^2}$$

$$T = \frac{PL}{2}\sqrt{1 + \frac{L^2}{16d^2}}$$

$$T = H\sqrt{1 + \frac{16d^2}{L}}$$

The inclination β of T with the vertical is given by

$$\tan\beta = \frac{H}{V} = \frac{PL^2}{8d} \cdot \frac{2}{PL} = \frac{L}{4d}$$

(c) Shape of the Cable

Let us ~~now~~ The shape of the cable under the uniformly distributed load.

from eq^n

$$H \cdot y = \frac{P \cdot L \cdot x}{2} - \frac{P \cdot x^2}{2}$$

$$H = \frac{PL^2}{8d}$$ Then

$$\left(\frac{PL^2}{8d}\right)y = \frac{PLx}{2} - \frac{Px^2}{2}$$

$$y = \frac{4dx}{L^2}(L - x)$$

This is, Thus the eq^n of the curve with respected to the cable chord. The Cable Thus, takes the form of parabola When subjected to uniformly distributed load.

(d) Length of the Cable :

(1) Both ends at the same Level

When ~~both~~ the ends of the cable are at the same level. the eq^n of Parabola can be written ; with C as the origin

$$y = kx^2$$

At A , $x = \frac{L}{2}$ and $y = d$

$$\therefore \quad k = \frac{y}{x^2} = \frac{d}{(L/2)^2} = \frac{4d}{L^2}$$

$$\therefore \quad y = \frac{4d}{L^2} x^2$$

$$\therefore \quad \frac{dy}{dx} = \frac{8d}{L^2} x.$$

Consider one element of length ds of the curve, having co-ordinates x and y. The Total length S of the curve is

$$S = \int_0^L ds = 2 \int_0^{L/2} \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2} dx$$

$$= 2 \int_0^{L/2} \left(1 + \frac{64d^2}{L^4} x^2\right)^{1/2} dx$$

By Binomial theorem

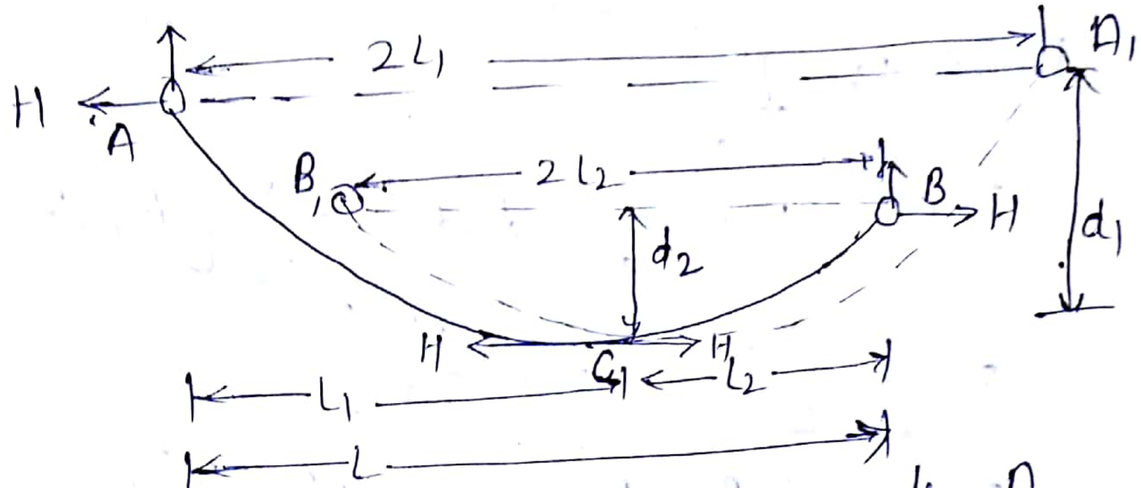
$$S = 2 \int_0^{L/2} \left(1 + \frac{1}{2} \frac{64d^2}{L^4} x^2 + \dots\right) dx$$

$$= 2 \left[x + \frac{32d^2}{3L^4} x^3\right]_0^{L/2} = 2\left[\frac{L}{2} + \frac{4}{3} \frac{d^2 L^3}{L^4}\right]$$

$$\boxed{S = L + \frac{8d^2}{3L}}$$

Length of the Cable:

Ends at different level



Consider a cable AB with the supports A and B at different Levels. Let C be the lowest point of the cable, such that the horizontal equivalent of AC is L_1 and that ~~of~~ CB is L_2

$$L = L_1 + L_2 \quad -(1)$$

$$H = \frac{P}{8}\frac{(2L_1)^2}{d_1} = \frac{PL_1^2}{2d_1} \quad -(2)$$

$$H = \frac{P}{8}\frac{(2L_2)^2}{8d_2} = \frac{PL_2^2}{2d_2} \quad -(3)$$

Since H is the same at C for both the portion of cable

$$\frac{PL_1^2}{2d_1} = \frac{PL_2^2}{2d_2}$$

$$\boxed{\frac{L_1}{L_2} = \sqrt{\frac{d_1}{d_2}}} \quad -(4)$$

from eq^n (1) and (4) the values of L_1 and L_2 can be known in terms of L, d_1 and d_2.

In order to find the vertical reaction V_A at A. Taking moment about B

$$V_A = \frac{1}{L}\left[\frac{PL^2}{2} + H(d_1 - d_2)\right]$$

Where $H = \frac{PL_1^2}{2d_1}$

$$= \frac{P}{2L}\left(L^2 + \frac{L_1^2}{d_1}(d_1 - d_2)\right) = \frac{P}{2L}\left[L^2 + L_1^2 - L_1^2\frac{d_2}{d_1}\right]$$

$$= \frac{P}{2L}\left[L^2 + L_1^2 - L_1^2 \times \frac{L_2^2}{L_1^2}\right]$$

$$= \frac{P}{2L}\left[L_1^2 + L_2^2 + 2L_1L_2 + L_1^2 - L_2^2\right]$$

$$= \frac{P}{2L}\left[2L_1^2 + 2L_1L_2\right]$$

$\therefore V_A = PL_1, \quad V_B = PL_2$

For imaginary cable ACA_1, the length

$$S_1 = 2L_1 + \frac{8}{3}\frac{d_1^2}{2L_1} = 2L_1 + \frac{4}{3}\frac{d_1^2}{L_1}$$

Similarly, the length of cable BCB_1

$$S_2 = 2L_2 + \frac{8}{3}\frac{d_2^2}{2L_2} = 2L_2 + \frac{4}{3}\frac{d_2^2}{L_2}$$

Hence the total length of the cable ABC is

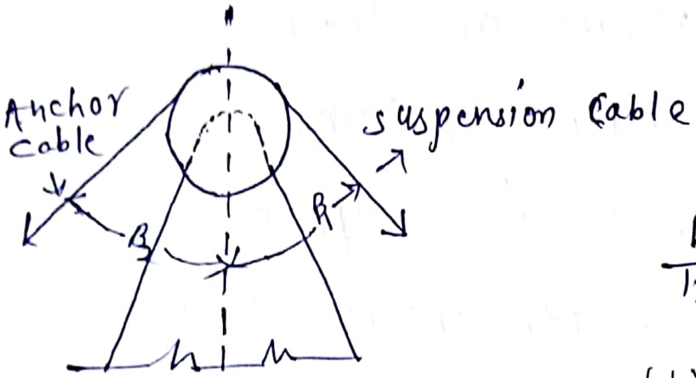
$$S = \frac{1}{2}(S_1 + S_2) = \frac{1}{2}\left\{\left(2L_1 + \frac{4}{3}\frac{d_1^2}{L_1}\right) + \left(2L_2 + \frac{4}{3}\frac{d_2^2}{L_2}\right)\right\}$$

$$S = L_1 + \frac{2}{3}\frac{d_1^2}{L_1} + L_2 + \frac{2}{3}\frac{d_2^2}{L_2}$$

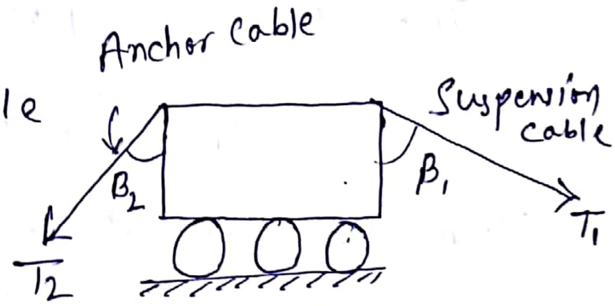
$$\boxed{S = L + \frac{2}{3}\frac{d_1^2}{L_1} + \frac{2}{3}\frac{d_2^2}{L_2}}$$

Anchor Cable ⇒

The suspension Cable is supported on either side, on supporting towers. The anchor cable transfer the tension of the suspension Cable to the anchorage Consisting of huge mass of concrete. The suspension Cable can either be passed two types of supports



(a) Guide pulley support



(b) Roller support

In case (a) when the suspension cable passes over the Guide pully and forms the part of the anchor Cable to the other side the tension T in the Cable is the same on the both side

β_1 = Inclination of the suspension cable with the vertical

β_2 = Inclination of the anchor cable with vertical

Pressure on the top of Pier

$$V_p = T\cos\beta_1 + T\cos\beta_2$$

$$= T(\cos\beta_1 + \cos\beta_2)$$

Horizontal force on the top of the Pier

$$= T\sin\beta_1 - T\sin\beta_2$$

$$= T(\sin\beta_1 - \sin\beta_2)$$

In case (b) the cable supported on a saddle mounted on rollers the horizontal components of the tension in the suspension cable and the anchor cable will be equal since the roller supports do not have any horizontal Reaction

$$\therefore \quad T_1 \sin\beta_1 = T_2 \sin\beta_2 = H$$

The vertical pressure on the pier

$$V_p = T_1 \cos\beta_1 + T_2 \cos\beta_2$$

Temperature stresses in suspension cable

Let s = length of the cable

δs = Change in the length due to change in temperature

δd = Corresponding change in dip

$$S = L + \frac{8}{3}\frac{d^2}{L}$$

$$\delta s = \frac{16}{3}\frac{d}{L}\delta d \quad \text{or} \quad \delta d = \frac{3}{16}\frac{L}{d}\delta s \quad -(1)$$

But $\delta s = s \cdot \alpha \cdot t$

α = Coefficient of thermal expansion of cable

t = Change in temperature

$$\delta s = \alpha t \left(L + \frac{8}{3}\frac{d^2}{L}\right)$$

$$\delta s = L\alpha t + \frac{8}{3}\frac{d^2}{L}\alpha t$$

Neglecting $\frac{8}{3}\frac{d^2}{L}\alpha t$ in comparison to $L \cdot \alpha \cdot t$

$$\delta s = L \cdot \alpha \cdot t \quad -(2)$$

Substituting this in eq (1)

$$\boxed{\delta d = \frac{3}{16}\frac{L}{d}(L \cdot \alpha \cdot t) = \frac{3}{16}\frac{L^2}{d}\alpha t}$$

When the temperature rises, L will increase, and hence δd will increase. Similarly when the temperature falls. L will decrease. and hence δd will decrease.

Let us now find the corresponding change in the value of H due to this change in d

$$H = \frac{Pl^2}{8d} \quad \text{or} \quad H \propto \frac{1}{d}$$

$$\frac{\delta H}{H} = -\frac{\delta d}{d}$$

If f is the stresses in the cable

$$\frac{\delta f}{f} = \frac{\delta H}{H} = -\frac{\delta d}{d}$$

δf = change in the cable stress

$$\frac{\delta f}{f} = \frac{\delta H}{H} = -\frac{\delta d}{d} = -\frac{3}{16}\frac{l^2}{d^2}\alpha \cdot t$$

Q1 find out the value of a property consisting of land and building from the following data

Rent inclusive of all taxes = Rs. 600 Per Month

including outgoings sinking fund = 20% of gross rent

Net yield expected from the Property = 6%

future life of building = 50 years

Q2 An Electric motor was purchased for Rs 15000/-. Assuming the life of the motor as 16 years and scrap value as 10% of the original cost. Calculate the book value after 10 years

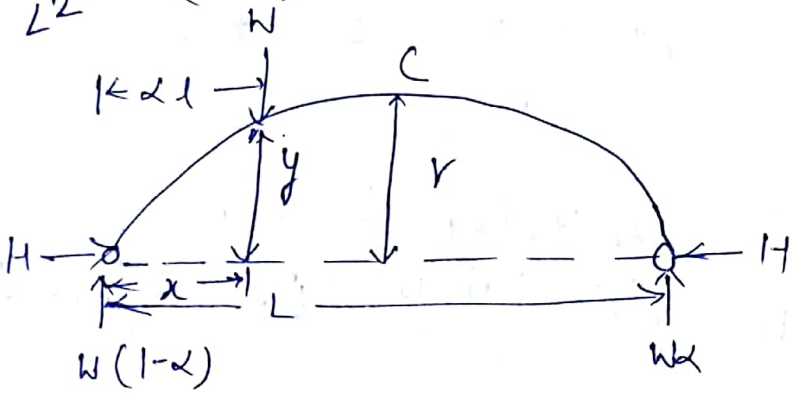
Two hinged Parabolic Arch: Expression for H

Consider a two hinged parabolic arch of horizontal span L and central rise r, subjected to a point load W at a distance ~~d~~ αl from the left.

The equation of arch is

$$y = \frac{4r}{L^2} x(L-x)$$

Now



Now
$$H = \frac{\int_0^L \mu y\,dx}{\int_0^L y^2\,dx} \quad -(1)$$

The numerator $= \int_0^L \mu y\,dx$

$$= \int_0^{\alpha L} \mu y\,dx + \int_{\alpha L}^{L} \mu y\,dx = a+b \quad -(2)$$

The quantity $a = \int_0^{\alpha L} \mu y\,dx = \int_0^{\alpha L} W(1-\alpha)x \cdot \frac{4r}{L^2} x(L-x)\,dx$

$$= \frac{(1-\alpha)4rW}{L^2}\left(\frac{Lx^3}{3} - \frac{x^4}{4}\right)_0^{\alpha L} = \frac{(1-\alpha)4rW}{L^2}\left(\frac{L^4\alpha^3}{3} - \frac{L^4\alpha^4}{4}\right) -(3)$$

The quantity

$$b = \int_{\alpha L}^{L} W\alpha(L-x) \cdot \frac{4r}{L^2} x(L-x)\,dx = \frac{4r\alpha W}{L^2}\int_{\alpha L}^{L}(L^2x + x^3 - 2Lx^2)\,dx$$

$$= \frac{4r\alpha W}{L^2}\left[\left(\frac{L^2 \cdot L^2}{2} + \frac{L^4}{4} - \frac{2L \cdot L^3}{3}\right) - \left(\frac{L^2 \cdot \alpha^2 \cdot L^2}{2} + \frac{\alpha^4 L^4}{4} - \frac{2L\alpha^3 L^3}{3}\right)\right]$$

$$= \frac{4r\alpha W L^4}{12L^2}\left(1 - 6\alpha^2 - 3\alpha^4 + 8\alpha^3\right) \quad -(4)$$

The numerator

$$= \left[\frac{(1-\alpha)4rW}{L^2}\left(\frac{L^4\alpha^3}{3}-\frac{\alpha^4L^4}{4}\right)+\frac{4r\alpha WL^2}{12}\left(1-6\alpha^2-3\alpha^4+8\alpha^3\right)\right] \quad - (5)$$

The denominator

$$= \int_0^L y^2\,dx = \frac{16r^2}{L^4}\int_0^L x^2(L-x)^2\,dx = \frac{16r^2}{L^4}\int_0^L (x^2L^2+x^4-2Lx^3)\,dx$$

$$= \frac{16r^2}{L^4}\left(\frac{L^5}{3}+\frac{L^5}{5}-\frac{L^5}{2}\right) = \frac{16r^2L}{30}(10+6-15)$$

$$= \frac{8}{15}r^2L \quad - (6)$$

eq substituting in eq (1)

$$H = \frac{\left[\frac{(1-\alpha)4rW}{L^2}\left(\frac{L^4\alpha^3}{3}-\frac{\alpha^4L^4}{4}\right)+\frac{4r\alpha}{12}WL^2(1-6\alpha^2-3\alpha^4-8\alpha^3)\right]}{\frac{8}{15}r^2L}$$

which reduces to

$$H = \frac{5}{8}W\frac{L}{r}\alpha(1-\alpha)(1+\alpha-\alpha^2)$$

ree Hinged Stiffening girder ⇒

The cable of the suspension bridge is the main load bearing member, the curvature of the cable of an unstiffned bridge change as the load moves on the decking. To avoid this, the decking is stiffened by provision of either a three hinged stiffening girder or a two hinged stiffening girder.

We shall now consider the effect of a unit point load on the decking

(i) bending moment diagram for fixed load position

(ii) Influence line for horizontal reaction H of the cable

(iii) Influence line for bending moment at a section.

(iv) Maximum bending moment diagram due to a point load W.

(v) Maximum bending moment diagram due to a uniformly distributed load of intensity w.

For the purpose of analysis of the above items. Let us consider the equilibrium of the cable as well as stiffening girder seperately.

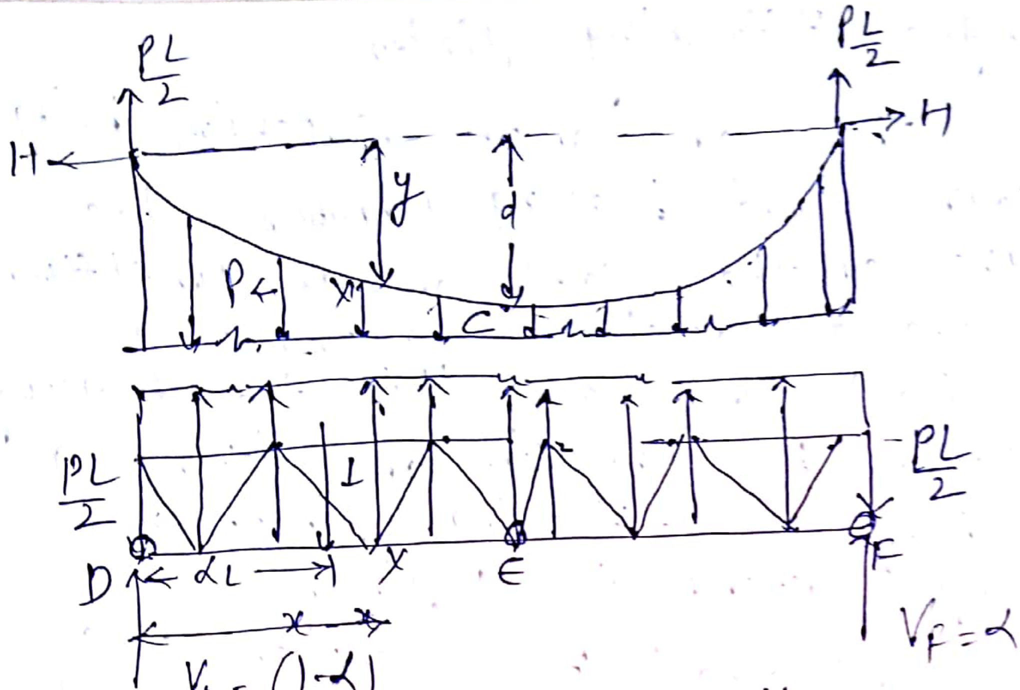
(i) Equilibrium of the cable

In upper part of fig

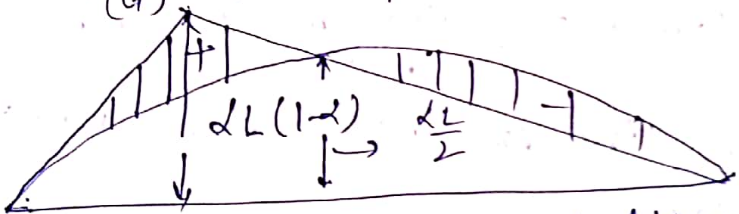
vertical Reaction equal to $\frac{PL}{2}$

and horizontal Reaction

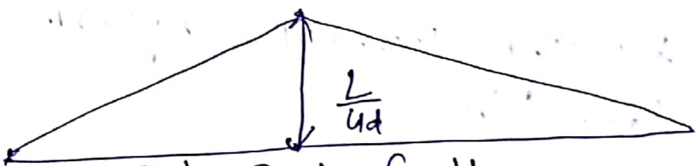
$$H = \frac{PL^2}{8d} \quad -(1)$$



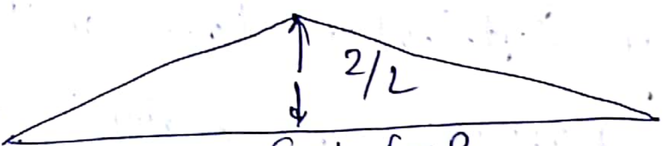
(a) The suspension bridge



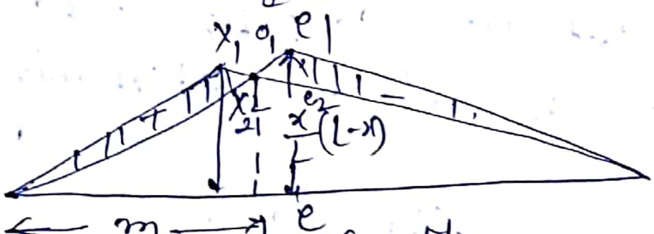
(b) Bending moment diagram



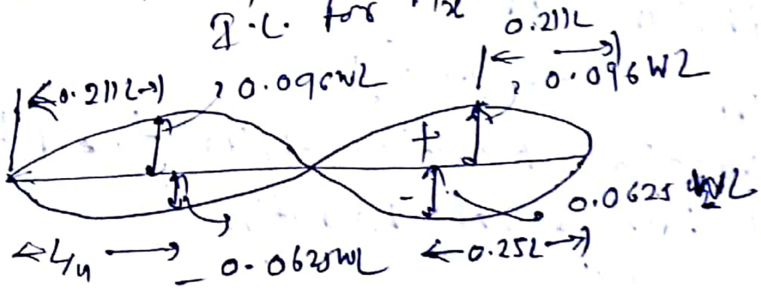
(c) I.L. for H



I.L. for P



I.L. for Mx



the cable is flexible, the bending moment at any point on it is equal to zero.

$$M_x = 0 = -Hy + \frac{PL}{2} \times x - \frac{Px^2}{2}$$

$$Hy = \frac{PL}{2} x - \frac{Px^2}{2} \quad -(2)$$

The equation of the parabola, with A as the origin

$$y = Kx\,(L-x)$$

At $x = \frac{L}{2}$, $y = d$ ⇒ $K = \frac{4d}{L^2}$

$$y = \frac{4dx}{L^2}(L-x) \quad -(3)$$

(2) Equilibrium of the Girder → for the equilibrium The lower part of the equilibrium for the three hinged stiffening girder which is subjected to following forces

(i) The External unit load

(ii) The simply supported Reaction $V_D = (1-\alpha)$ and $V_F = \alpha$ respectively at point D and F.

(iii) The uniformly distributed Pull P exerted by suspenders

(iv) Downward reaction $\frac{PL}{2}$ at D and F due to the Pull P of the hangers

(3) Bending Moment diagram.

$$M_x = [+V_D \cdot x - 1(x-\alpha L)] + \left[-\frac{PL}{2}x + \frac{Px^2}{2}\right]$$

$$= \mu_x - Hy \quad -(4)$$

The first part that is $[+V_D \cdot x - 1(x-\alpha L)]$ may be designated as μ_x. Where μ_x is the bending moment at point x treating the girder as simply supported beam.

The μ_x diagram for a simply supported beam is triangle having an ordinate $\frac{Wab}{L} = \frac{1 \times \alpha L(1-\alpha)L}{L} = \alpha L(1-\alpha)$

Hy diagram will be Parabola.

$$H = \frac{\alpha L}{2d}$$

The maximum value of y is equal to d at the centre.

The Hy diagram is a parabola having maximum ordinate of $\frac{\alpha L}{2d} \cdot d = \frac{\alpha L}{2}$ under the centre of the cable.

(4) Influence line for H

$$M_E = 0 = M_E - Hy \quad \text{or} \quad H = \frac{M_E}{y}$$

$$M_E = \frac{\alpha L}{2}, \quad y = d \qquad \alpha L$$

$$H = \frac{\alpha L}{2d} \qquad \text{-(5)}$$

When $\alpha L = 0$, $H = 0$: when $\alpha L = \frac{L}{2}$; $H = \frac{L}{4d}$

(5) Influence line for P.

$$H = \frac{PL^2}{8d} \quad \text{or} \quad P = \frac{8d}{L^2} \cdot H$$

$$H = \frac{\alpha L}{2d} \Rightarrow P = \frac{8d}{L^2} \cdot \frac{\alpha L}{2d} = \frac{4\alpha}{L} \quad \text{-(6)}$$

It valid for $\alpha L = 0$ to $\alpha L = \frac{L}{2}$

When $\alpha L = 0$ $\therefore P = 0$

$\alpha L = \frac{L}{2}$ or $\alpha = \frac{1}{2}$ $\Rightarrow P = \frac{4}{L} \cdot \frac{1}{2} = \frac{2}{L}$

(6) Influence line for Bending Moment

$$M_x = M_x - Hy.$$

The I.L for M_x is a triangle having a maximum ordinate of $\frac{x(L-x)}{L}$ under the section X. The I.L for $-Hy$ will be triangle.

Thus the I.L for $-Hy$ has maximum ordinate principal centroidal axes

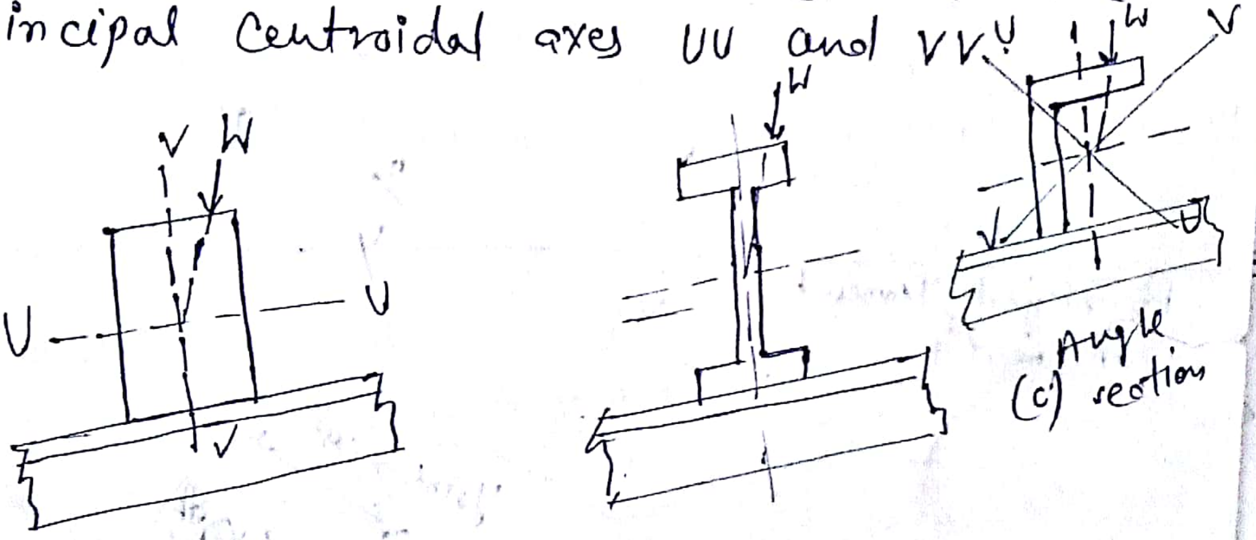
Unit-IV

Unsymmetrical Bending

Unsymmetrical Bending ⇒

The Plane of loading or that of bending does not lie in a plane that contains the principal centroidal axes of the cross-section, the bending is called unsymmetrical bending.

Some cases of unsymmetrical bending in which the plane of load W is vertical and does not concide with the Principal Centroidal axes UU and VV.

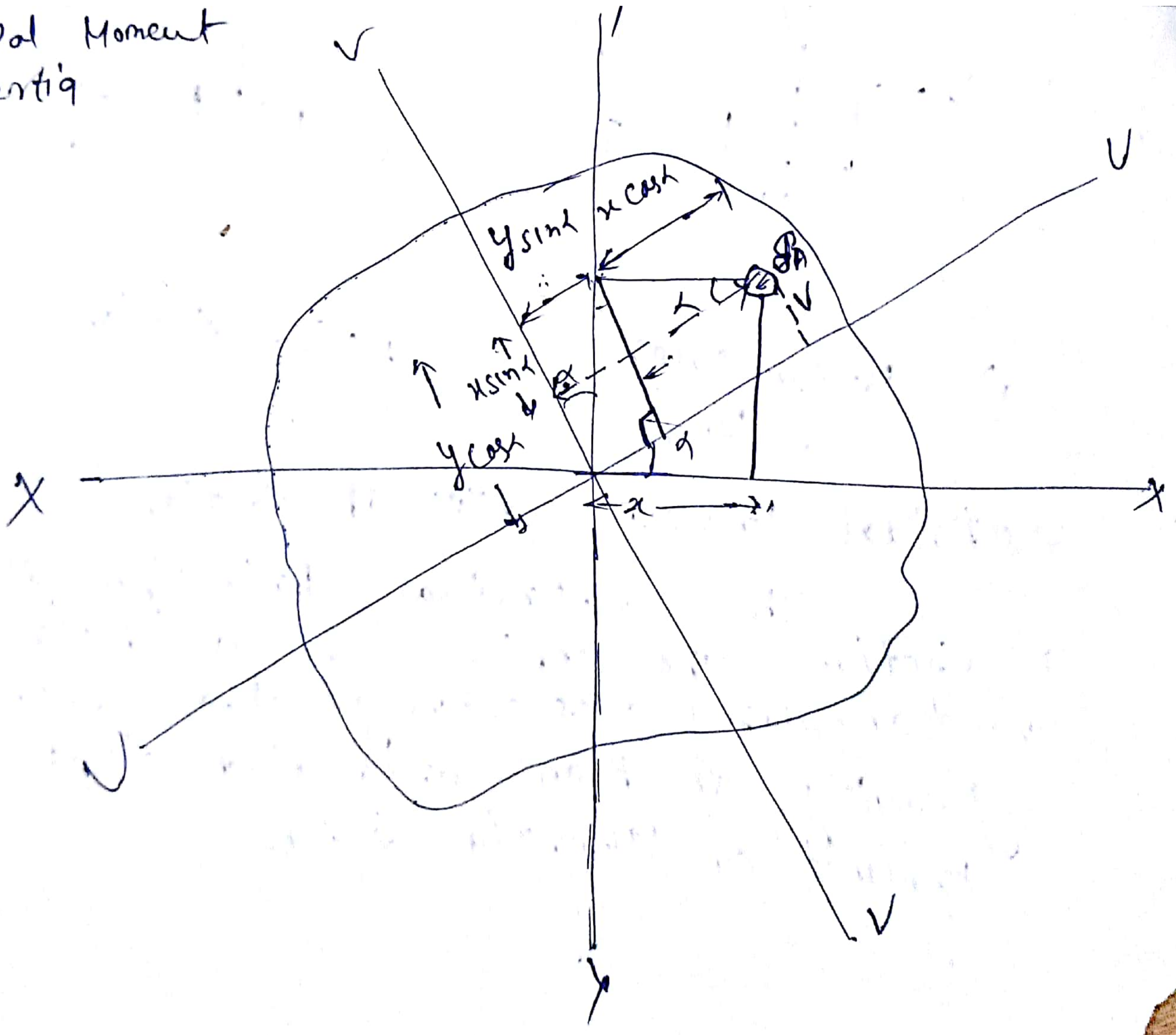


(a) Rectangular section (b) I- section (c) Angle section

Centroidal Principal Axes of section.

The Centroidal Principal axes of a section are defined as a pair of rectangular axes through the centre of gravity of plane area such that the product of Inertia is zero.

Principal Moment of Inertia



Let U-U, V-V = Principal centroidal axes

X-X, Y-Y = Any pair of centroidal Rectangular axes.

α = angle b/w U-U and X-X axes.

If the U-U, V-V are the Principal axes, the Product of inertia $\Sigma u \cdot v \cdot \delta a = 0$ where δa is an elementary area with co-ordinates u and v referred to the co-ordinates axes.

Let x, y be co-ordinates of an elementary area δa with respect to the x-y axes.

By definition

$$I_{xx} = \Sigma y^2 \delta a, \quad I_{yy} = \Sigma x^2 \delta a;$$

$$I_{xy} = \Sigma xy \, \delta a$$

$$\& \; I_{UU} = \Sigma v^2 \delta a, \quad I_{VV} = \Sigma u^2 \delta a, \quad I_{UV} = \Sigma uv \delta a$$

The relationship b/w x, y, and u, v co-ordinates

$$u = x\cos\alpha + y\sin\alpha$$

$$v = y\cos\alpha - x\sin\alpha$$

Hence $I_{UU} = \Sigma v^2 \delta a = \Sigma (y\cos\alpha - x\sin\alpha)^2 \delta a$

$$= \cos^2\alpha \, \Sigma y^2 \delta a + \sin^2\alpha \, \Sigma x^2 \delta a - 2\sin\alpha\cos\alpha \, \Sigma xy \, \delta a$$

$$= I_{xx} \cos^2\alpha + I_{yy} \sin^2\alpha - I_{xy} \sin 2\alpha \quad -(1)$$

$$I_{VV} = \Sigma u^2 \delta a = \Sigma (x\cos\alpha + y\sin\alpha)^2 \delta a$$

$$= I_{xx} \sin^2\alpha + I_{yy} \cos^2\alpha + I_{xy} \sin 2\alpha \quad -(2)$$

$$I_{UV} = \Sigma uv \, \delta a = \Sigma (x\cos\alpha + y\sin\alpha)(y\cos\alpha - x\sin\alpha)\delta a$$

$$= \cos^2\alpha \, \Sigma xy \, \delta a - \sin^2\alpha . \Sigma xy \, \delta a + \sin\alpha\cos\alpha \left(\Sigma y^2 \delta a - \Sigma x^2 \delta a\right)$$

$$= \cos^2\alpha . I_{xy} - \sin^2\alpha \, I_{xy} + \sin\alpha . \cos\alpha \, (I_{xx} - I_{yy})$$

$$= \left(\frac{I_{xx} - I_{yy}}{2}\right) \sin 2\alpha + I_{xy} \cos 2\alpha$$

Since U-U and V-V are the Principal axes

$$I_{UV} = 0 = \left(\frac{I_{xx} - I_{yy}}{2}\right) \sin 2\alpha + I_{xy} \cos 2\alpha$$

$$\tan 2\alpha = -\frac{2 I_{xy}}{I_{xx} - I_{yy}}$$

Knowing I_{xx}, I_{yy}, I_{xy} the angle α can be calculated.

Substituting α in eq. the moment of inertia about the principal axes can be determined.

Analytical solution

$$I_{UU} = \frac{I_{xx} + I_{yy}}{2} + \frac{I_{xx} - I_{yy}}{2} \cos 2\alpha - I_{xy} \sin 2\alpha$$

$$I_{VV} = \frac{I_{xx} + I_{yy}}{2} - \frac{I_{xx} - I_{yy}}{2} \cos 2\alpha + I_{xy} \sin 2\alpha$$

$$\sin 2\alpha = \frac{-I_{xy}}{\sqrt{\left(\frac{I_{xx} - I_{yy}}{2}\right)^2 + I_{xy}^2}}$$

$$\cos 2\alpha = \frac{I_{xx} - I_{yy}}{2\sqrt{\left(\frac{I_{xx} - I_{yy}}{2}\right)^2 + I_{xy}^2}}$$

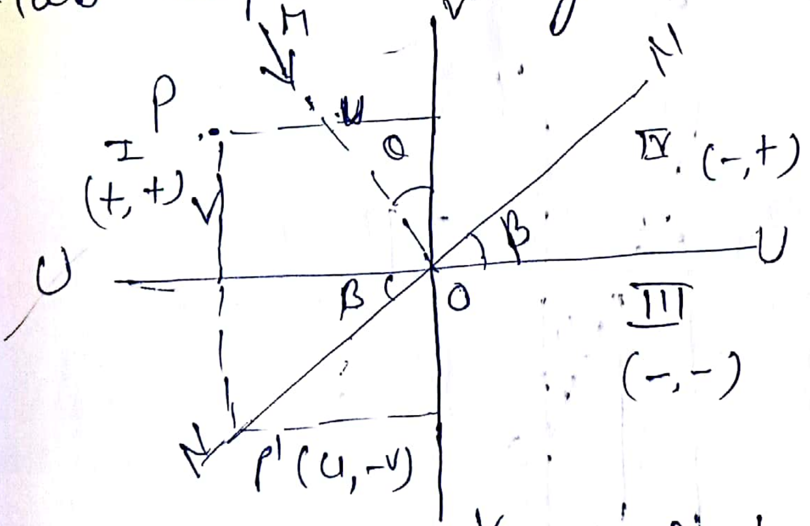
$$I_{UU} = \frac{I_{xx} + I_{yy}}{2} + \sqrt{\left(\frac{I_{xx} - I_{yy}}{2}\right)^2 + I_{xy}^2}$$

$$I_{VV} = \frac{I_{xx} + I_{yy}}{2} - \sqrt{\left(\frac{I_{xx} - I_{yy}}{2}\right)^2 + I_{xy}^2}$$

Bending stress in beam subjected to unsymmetrical Bending ⇒

In case of simple bending where the plane of loading coincides with one of the principal plane, the neutral axis the perpendicular to the principal plane and passes through the centroid of section. In case of unsymmetrical bending ~~neutral~~ Neutral axis is not perpendicular to the plane of bending. The bending stress at any point in the beam subjected to unsymmetrical bending can be determined by following method.

Resolution of bending moment into two components along principal axes



Let the plane of bending (M) be inclined at an angle θ with one of the principal planes.

The intensity of bending stress at any point P(u,v) will be algebraic sum of the stress due to the components bending moment.

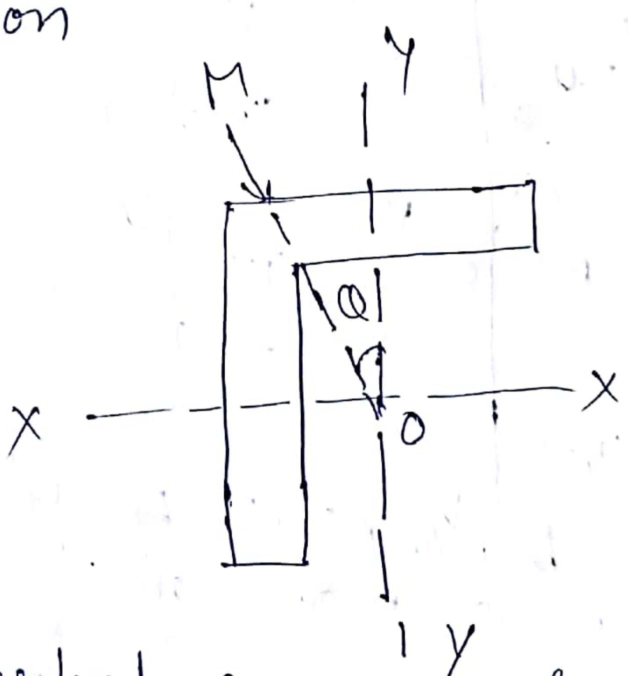
The final bending stress at P.

$$\sigma_b = \frac{M\cos\theta}{I_{UU}} V + \frac{M\sin\theta}{I_{VV}} \cdot U$$

The Method is suitable to those sections which have at least one axis of symmetry which is also of principal axes.

(ii) Resolution of B.M into Any two Rectangular Axes through the centroid.

The Most generally method of finding the bending stress at any point is to resolve M along any two rectangular axes passing through the centroid of the section



The resolved component of M along the Y-Y axis is designated as M_{xx} and is equal to $M\cos\theta$; similarly resolved component of M along X-X axis is designated as M_{yy}

equal to $M \sin\alpha$

The bending stress f_b at any point $P(x, y)$

$$f_b = a_1 x + b_1 y$$

Now M_{xx} = bending moment about X-axis

$$= \int f_b \cdot \delta a \cdot y = \int (a_1 x + b_1 y)\, y\, \delta a$$

$$= a_1 \int xy\, \delta a + b_1 \int y^2 \delta a$$

$$= a_1 I_{xy} + b_1 I_{xx} \quad \text{(1)}$$

M_{yy} = bending moment about Y-axis

$$= \int f_b \cdot \delta a \cdot x = \int (a_1 x + b_1 y)\, x\, \delta a$$

$$= a_1 \int x^2 \delta a + b_1 \int xy\, \delta a$$

$$= a_1 \cdot I_{yy} + b_1 \cdot I_{xy} \quad \text{(11)}$$

Using eq (1) & (2)

$$a_1 = \frac{M_{yy} \cdot I_{xx} - M_{xx} \cdot I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}$$

$$b_1 = \frac{M_{xx} \cdot I_{yy} - M_{yy} \cdot I_{xy}}{I_{xx} \cdot I_{yy} - I_{xy}^2}$$

$$f_b = \frac{M_{yy} I_{xx} - M_{xx} I_{xy}}{I_{xx} \cdot I_{yy} - I_{xy}^2} \cdot x + \frac{M_{xx} I_{yy} - M_{yy} I_{xy}}{I_{xx} I_{yy} - I_{xy}^2} \cdot y$$

(iii) Location of Neutral Axis

In the case of unsymmetrical bending the neutral axis is neither perpendicular to the plane of bending, nor perpendicular to any of the principal planes.

θ = Inclination of the plane of bending to the V-V axis

β = Inclination of neutral axis with the U-U axis

At any point P on it, the bending stress is equal to zero.

$$f_b = 0 = \frac{M\cos\theta}{I_{vv}} \cdot v + \frac{M\sin\theta}{I_{vv}} u$$

$$v = -u \frac{I_{uu}}{I_{vv}} \tan\theta$$

It is equation of the neutral axis N-N which is a straight line.

It is clear that $v = 0, u = 0$ ∴ hence the neutral axis passes through the centroid of the section.

$$\tan\beta = -\frac{v}{u}$$

$$-\frac{v}{u} = \frac{I_{uu}}{I_{vv}} \tan\theta$$

Hence $\boxed{\tan\beta = \frac{I_{uu}}{I_{vv}} \tan\theta}$

Thus the neutral axis located from this eq.

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I_{NN} = Moment of inertia of the beam about neutral axis

$$I_{NN} = I_{UU}\cos^2\beta + I_{VV}\sin^2\beta$$

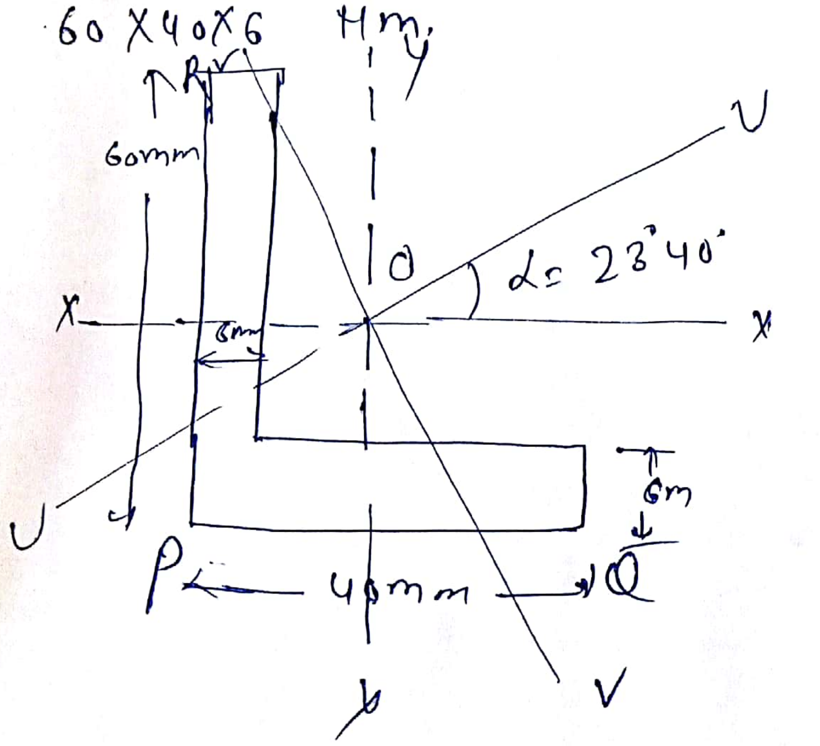
The plane of loading is inclined at angle $(90-\theta+\beta)$ with the N.A. If the line is drawn perpendicular to the neutral axis, the plane of bending will be inclined at $(\beta-\theta)$ to the line. Hence component of bending moment along the axis

$$M_{NN} = M\cos(\beta-\theta)$$

y_N = Perpendicular distance of any point from the neutral axis

$$\boxed{f_b = \frac{M\cos(\beta-\theta)}{I_{NN}} \cdot y_N}$$

Determine the principal moment of inertia for an unequal angle section 60 X 40 X 6 mm.



Sol^{n} Let O be the centroid of the section and let X ~~axis~~ ~~and~~ ~~cg~~ be at a distance C_x from face PQ, and Y axis be a distance C_y from face PR.

$$A = A_1 + A_2 = (40 \times 6) + (54 \times 6) = 240 + 324$$

$$= 564 \text{ mm}^2$$

$$Q_x = 40 \times 6 \times$$

Give the Expression for length of the Cable
(I) Both Ends at the same level
(II) When Both Ends is different level

Q2 Give the General Cable theorem and Temperature stresses in suspension cable

Q3 A wire of uniform material weighing 0.50 lb. per cu. inch hangs b/w two points 150 ft. appart horizontally, with one end 5 ft above the other. The sag of the wire measured from the highest point is 9 ft. Calculate the maximum stress in the wire.

or

Q4 The three hinged stiffening girder of a suspension bridge of 100 m span is subjected to two point loads of 10 kN each placed at 20 m and 40 m respectively from the left hand hinge. Determine the B.M and S.F in the girder at section 30 m from each end. and Maximum tension in the cable which has a central dip of 10 m.

Q5 Explain the theories of failure
or

Q6 Write down the short note on following terms
(I) Unsymmetrical bending
(II) Centroidal Principal axes.
(III) Shear Centre
(IV) location of neutral axis

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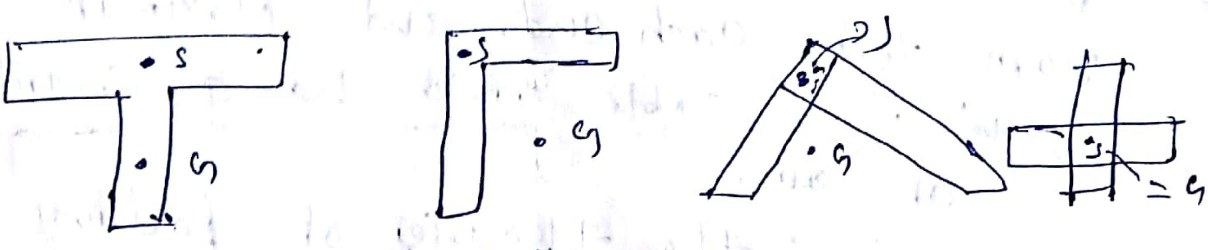
Q4 (5, 94, 303) A beam of Rectangular section 100 mm wide and 180 mm deep is subjected to a bending Moment of 20 kN-m. The Trace of the plane of loading is inclined at 45° to the Y-Y axis of the section. Locate the Netural axis of the section and calculate the maximum bending stress induced in the section.

Q5. Determine the principal moment of inertia for an unequal angle section 60X40X6 mm.

Shear Centre

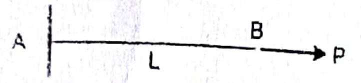
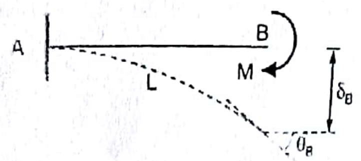
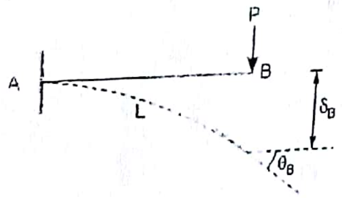
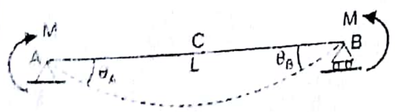
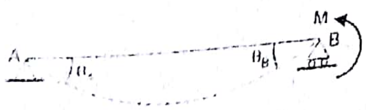
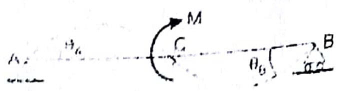
Shear centre is a point which a Concentrated load passes then there will be only bending and no twisting. It is also called bending centre. It is that point of shear centre through which the resultant shear passes. Shear centre always lies on the axis of the symmetry if exists

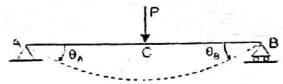


s = shear centre
G = centre of gravity

Flexibility Matrix, and

10.4 Standard Results of Slope and Deflection

S.No.	Loading	Slope	Deflection
1	Axial load at free end of cantilever: 	$\theta_B = 0$	$\delta_B = \frac{PL}{AE}$ (Axial)
2	Moment at free end of cantilever: 	$\theta_B = \frac{ML}{EI}$	$\delta_B = \frac{ML^2}{2EI}$
3	Point load at free end of cantilever 	$\theta_B = \frac{PL^2}{2EI}$	$\delta_B = \frac{PL^3}{3EI}$
4	Simply supported beam with moment at both end 	$\theta_A = \theta_B = \frac{ML}{2EI}$	$\delta_C = \frac{ML^2}{8EI}$
5	Simply supported beam with moment at one end. 	$\theta_A = \frac{ML}{6EI}$ $\theta_B = \frac{ML}{3EI}$	$\delta_A = 0$ $\delta_B = 0$
6	Simply supported beam with moment at mid span: 	$\theta_A = \theta_B = \frac{ML}{24EI}$ $\theta_C = \frac{ML}{12EI}$	$\delta_A = 0$ $\delta_C = 0$ $\delta_B = 0$

7	Propped cantilever subjected to moment at propped support 	$\theta_E = \dfrac{ML}{4EI}$	$\delta_A = 0$ $\delta_B = 0$
8		$\overline{M}_{AB} = \overline{M}_{BA} = \dfrac{6EI\Delta}{L^2}$	$\delta_A = 0$ $\delta_B = \Delta$
9	Simply supported beam subjected to point load at mid span 	$\theta_A = \theta_B = \dfrac{PL^2}{16EI}$	$\delta_A = \delta_A = 0$ $\delta_C = \dfrac{PL^3}{48EI}$

10.5 Flexibility and Stiffness

$$\Delta = \frac{P}{K}$$

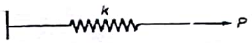


Fig.10.1

The displacement caused by unit force is known as **flexibility**

$$f = \frac{\Delta}{P}$$

The force required to produce unit displacement is known as **stiffness**

$$K = \frac{P}{\Delta}$$

Also note that, $\quad K = \dfrac{1}{f}$ or $K \times f = 1$

10.6 Flexibility Matrix

10.6.1 Properties

1. The flexibility matrix will always be a square matrix ($n \times n$) in which diagonal elements will be non-negative and non-zero.
2. Order of flexibility matrix will be equal to degree of static indeterminacy (i.e., no. of redundants)

10.6.2 Procedure to Develop Flexibility Matrix

If there are N coordinate then flexibility matrix will be $N \times N$ size square matrix. The element of flexibility matrix represents displacement of any point produced by unit force in the direction of chosen coordinate. Consider a cantilever beam, the chosen coordinates are shown in figure 10.2

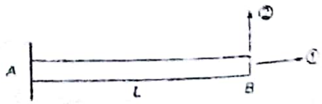


Fig. 10.2

There are two coordinates, therefore flexibility matrix will be of square size [2 × 2]

$$[f] = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$$

Note that in typical element of flexibility matrix i.e f_{ij}

j – represents direction of applied unit force

i – represents direction of displacement measured.

Here,

f_{11} = Displacement in direction of (1) when unit force is applied in the direction of (1) alone

f_{12} = Displacement in direction of (1) when unit force is applied in the direction of (2) alone

f_{21} = Displacement in direction of (2) when unit force is applied in the direction of (1) alone

f_{22} = Displacement in direction of (2) when unit force is applied in the direction of (2) alone

According to Maxwell's reciprocal theorem,

$$f_{ij} = f_{ji}$$

Hence,
$$f_{12} = f_{21}$$

Step-1. To generate first column of flexibility matrix apply unit force in the direction of coordinate (1) alone. Now measure the displacements produce in the directions of coordinate 1, 2, ... *N*.

Step-2. To generate second column of flexibility matrix now apply unit force in the direction of coordinate (2) alone and measure the displacements produce in the directions of coordinate 1, 2,.....*N*.

For given cantilever, apply unit load in direction of coordinate (1).

f_{11} = Displacement of point *B* in the direction of coordinate (1) due to unit force in the direction of (1) alone

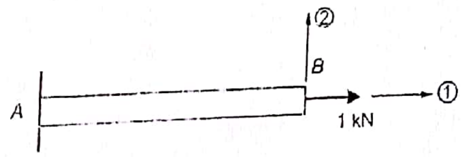


Fig. 10.3

$$\therefore \quad f_{11} = \frac{1 \times L}{AE} = \frac{L}{AE}$$

f_{21} = Displacement of point *B* in the direction of coordinate (2) due to unit force in the direction (1) alone

$$\therefore \quad f_{21} = 0$$

Also from Maxwell's reciprocal theorem

$$f_{12} = f_{21} = 0$$

Now apply unit force in the direction of coordinate (2) alone.

f_{22} = Displacement of point *B* in the direction of coordinate (2) due to unit force in the direction of (2) alone

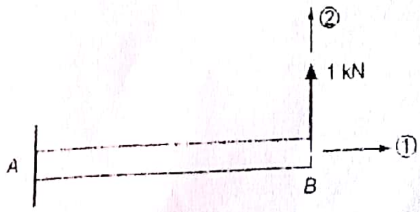


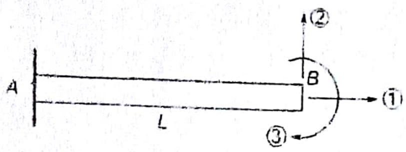
Fig. 10.4

$$f_{22} = \frac{1 \times L^3}{3EI} = \frac{L^3}{3EI}$$

Therefore flexibility matrix for given coordinate system is

$$[f] = \begin{bmatrix} \frac{L}{AE} & 0 \\ 0 & \frac{L^3}{3EI} \end{bmatrix}$$

Example 10.31 For the coordinate marked in figure, develop flexibility matrix.



Solution:

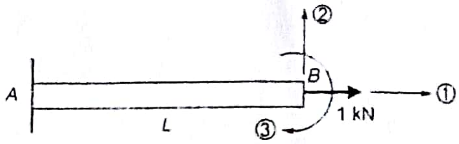
First Column:

Apply unit force in the direction of coordinate (1) alone and measure displacements in the direction of (1), (2) and (3)

$$f_{11} = \frac{1 \times L}{AE} = \frac{L}{AE}$$

$$f_{21} = 0$$

$$f_{31} = 0$$



Also from the Maxwell's reciprocal theorem,

$$f_{12} = f_{21} = 0$$

and $$f_{13} = f_{31} = 0$$

$\frac{ML^2}{2EI}$

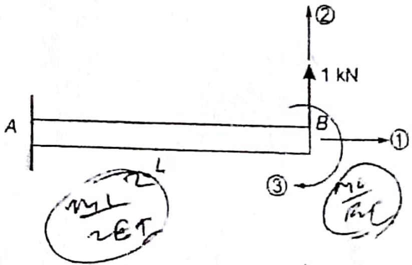
Second Column:

Apply unit force in the direction of coordinate (2) alone and measure displacements in direction of (1), (2) and (3)

$f_{12} = 0$ (already known)

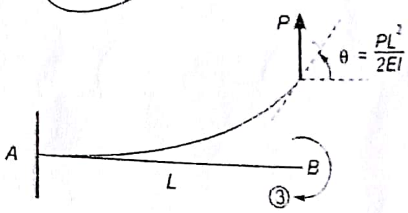
$$f_{22} = \frac{1 \times L^3}{3EI} = \frac{L^3}{3EI}$$

$$f_{32} = -\frac{1 \times L^2}{2EI} = -\frac{L^2}{2EI}$$



Also from the Maxwell's reciprocal theorem,

$$f_{23} = f_{32} = -\frac{L^2}{2EI}$$



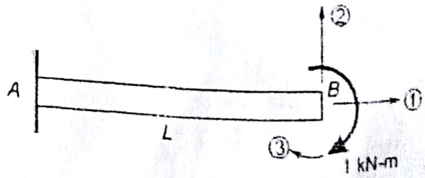
Third Column:

Apply unit moment in the direction of coordinate (3) alone and measure displacements in the directions of (1), (2) and (3)

$f_{13} = 0$ (Already known)

$f_{23} = -\frac{L^2}{2EI}$ (Already known)

$$f_{33} = \frac{1 \times L}{EI} = \frac{L}{EI}$$



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Therefore the flexibility matrix for given coordinate system is

$$[f] = \begin{bmatrix} \frac{L}{AE} & 0 & 0 \\ 0 & \frac{L^3}{3EI} & -\frac{L^2}{2EI} \\ 0 & -\frac{L^2}{2EI} & \frac{L}{EI} \end{bmatrix}$$

Example 10.4 Flexibility matrix for a beam element is written in the form:

$$[A] = \frac{L^3}{6EI}\begin{bmatrix} 2 & 5 \\ 5 & 16 \end{bmatrix}$$

What is the corresponding stiffness matrix?

(a) $\frac{6EI}{L^3}\begin{bmatrix} 16 & 5 \\ 5 & 2 \end{bmatrix}$ (b) $\frac{6EI}{7L^3}\begin{bmatrix} 16 & 5 \\ 5 & 2 \end{bmatrix}$

(c) $\frac{6EI}{L^3}\begin{bmatrix} 16 & -5 \\ -5 & 2 \end{bmatrix}$ (d) $\frac{6EI}{7L^3}\begin{bmatrix} 16 & -5 \\ -5 & 2 \end{bmatrix}$

Ans. (d)

Product of flexibility and stiffness matrix is an identity matrix i.e. flexibility matrix and stiffness matrix are inverse of each other.

$$[f][k] = [I]$$
$$[k] = [f]^{-1}$$

We know, for square matrix,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The inverse of [A] is given by,

$$[A]^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Similarly,

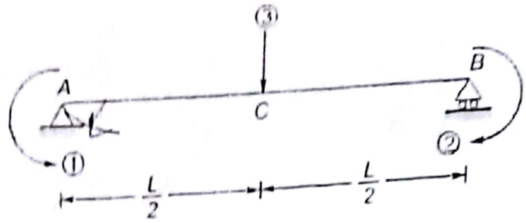
$$[f]^{-1} = \frac{1}{|f|}\frac{6EI}{L^3}\begin{bmatrix} 16 & -5 \\ -5 & 2 \end{bmatrix}$$

$$[k] = [f]^{-1} = \frac{6EI}{|f|L^3}\begin{bmatrix} 16 & -5 \\ -5 & 2 \end{bmatrix}$$

$$|f| = 2 \times 16 - 5 \times 5 = 7$$

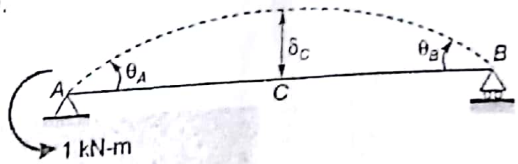
$$[k] = \frac{6EI}{7L^3}\begin{bmatrix} 16 & -5 \\ -5 & 2 \end{bmatrix}$$

Example 10.5 For the beam with coordinate shown in figure. Develop the flexibility matrix. EI is constant.



Solution:

First column: Apply unit moment at A in direction of (1).

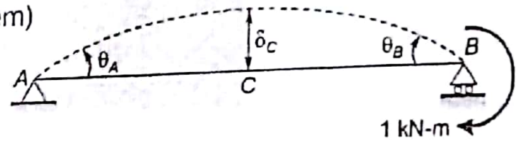


$$f_{11} = \frac{L}{3EI}$$

$$f_{21} = \frac{L}{6EI}$$

$f_{31} = f_{13}$ (By Reciprocal theorem)

Second column: Apply unit moment at B in direction of coordinate (2).

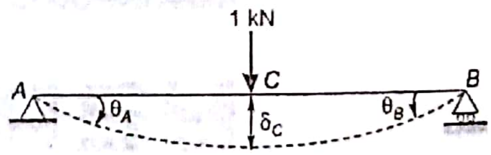


$f_{12} = f_{21} = \dfrac{L}{6EI}$ (By Reciprocal theorem)

$$f_{22} = \frac{L}{3EI}$$

$f_{32} = f_{23}$ (By Reciprocal theorem)

Third column: Apply unit load at C in direction of coordinate (3).



$$f_{13} = -\frac{L^2}{16EI}$$

$$f_{23} = -\frac{L^2}{16EI}$$

$$f_{33} = \frac{L^3}{48EI}$$

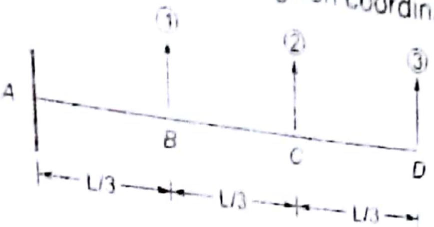
Hence, $$f_{31} = f_{13} = -\frac{L^2}{16EI}$$

$$f_{32} = f_{23} = -\frac{L^2}{16EI}$$

Therefore the flexibility matrix for given coordinate system is

$$[f] = \begin{bmatrix} \dfrac{L}{3EI} & \dfrac{L}{6EI} & \dfrac{-L^2}{16EI} \\ \dfrac{L}{6EI} & \dfrac{L}{3EI} & \dfrac{-L^2}{16EI} \\ \dfrac{-L^2}{16EI} & \dfrac{-L^2}{16EI} & \dfrac{L^3}{48EI} \end{bmatrix}$$

Example 10.6 Generate flexibility matrix for given coordinate system.



Solution:

First column: Apply unit load at B in the direction of coordinate (1) and measure displacements in the direction of 1, 2 and 3

$$f_{11} = \frac{\left(\frac{L}{3}\right)^3}{3EI} = \frac{L^3}{81EI}$$

$$f_{21} = \delta_B + \theta_B \cdot L_{BC}$$

$$f_{21} = \frac{L^3}{81EI} + \frac{(L/3)^2}{2EI} \times \frac{L}{3} = \frac{5}{162}\frac{L^3}{EI}$$

$$f_{31} = \delta_D$$

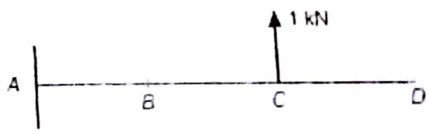
$$f_{31} = \delta_B + \theta_B \cdot L_{BD}$$

$$f_{31} = \frac{L^3}{81EI} + \frac{(L/3)^2}{2EI} \times \frac{2L}{3} = \frac{4}{81}\frac{L^3}{EI}$$

Second column: Apply unit load at C in the direction of coordinate (2) and measure displacements in the direction of 1, 2 and 3

$$f_{12} = f_{21} = \frac{5}{162}\frac{L^3}{EI} \text{ (By reciprocal theorem)}$$

$$f_{22} = \frac{1\left(\frac{2L}{3}\right)^3}{3EI} = \frac{8}{81}\frac{L^3}{EI}$$

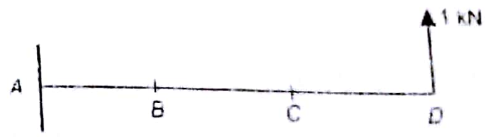


$$f_{32} = \delta_D = \delta_C + \theta_C \times L_{CD}$$

$$f_{32} = \frac{8}{81}\frac{L^3}{EI} + \frac{\left(\frac{2L}{3}\right)^2}{2EI} \times \frac{L}{3} = \frac{14}{81}\frac{L^3}{EI}$$

Third column: Apply unit load at D in the direction of coordinate (3) and measure displacements in the direction of 2 and 3

$$f_{13} = f_{31} = \frac{4}{81}\frac{L^3}{EI} \text{ (By reciprocal theorem)}$$



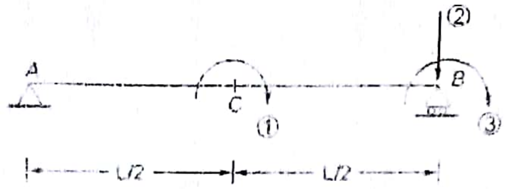
$$f_{23} = f_{32} = \frac{14}{81}\frac{L^3}{EI} \text{ (By reciprocal theorem)}$$

$$f_{33} = \frac{L^3}{3EI}$$

Hence for the given coordinate system the flexibility matrix is

$$[F] = \begin{bmatrix} \frac{L^3}{81EI} & \frac{5}{162}\frac{L^3}{EI} & \frac{4}{81}\frac{L^3}{EI} \\ \frac{5}{162}\frac{L^3}{EI} & \frac{8}{81}\frac{L^3}{EI} & \frac{14}{81}\frac{L^3}{EI} \\ \frac{4}{81}\frac{L^3}{EI} & \frac{14}{81}\frac{L^3}{EI} & \frac{L^3}{3EI} \end{bmatrix}$$

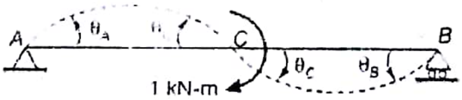
Example 10.7 Develop the flexibility matrix for the simply supported beam AB with coordinate system shown in figure.



Solution:

First column: Apply unit moment in the direction of coordinate (1) and measure displacements in the direction of 1, 2 and 3

$$f_{11} = \frac{1 \times L}{12EI} = \frac{L}{12EI}$$



$$f_{21} = 0$$

$$f_{31} = -\frac{1 \times L}{24EI} = \frac{-L}{24EI}$$

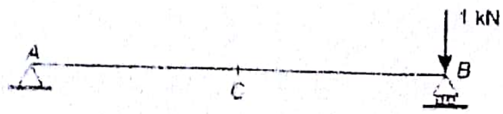
Also, from the Maxwell's reciprocal theorem

$$f_{12} = f_{21} = 0$$

$$f_{13} = f_{31} = -\frac{L}{24EI}$$

Second column: Apply unit load in the direction of coordinate and measure displacements in the direction of 1, 2 and 3

$$f_{12} = 0 \text{ (Already known)}$$



$$f_{22} = 0$$

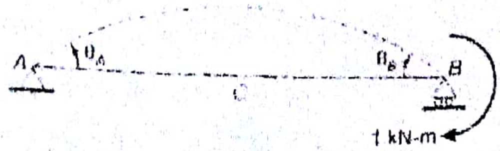
$$f_{32} = 0$$

Also, from Maxwell's reciprocal theorem,

$$f_{23} = f_{32} = 0$$

Third column: Apply unit moment in the direction of coordinate (3) and measure displacements in the direction of 1, 2 and 3

$$f_{13} = -\frac{L}{24EI} \text{ (Already known)}$$



$$f_{23} = 0 \text{ (Already known)}$$

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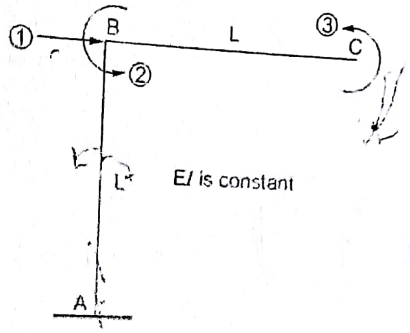
$$f_{33} = \frac{1 \times L}{3EI} = \frac{L}{3EI}$$

Hence the flexibility matrix for the coordinate system is

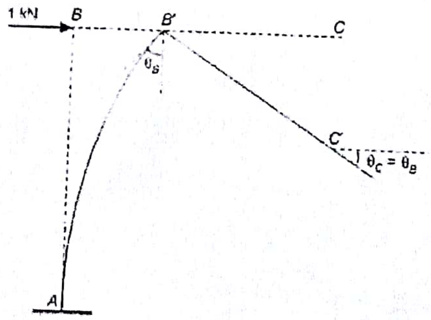
$$[f] = \begin{bmatrix} \frac{L}{12EI} & 0 & -\frac{L}{24EI} \\ 0 & 0 & 0 \\ -\frac{L}{24EI} & 0 & \frac{L}{3EI} \end{bmatrix}$$

Example 10.8 Generate flexibility matrix for cantilever frame shown in figure.



Solution:

First column:



$$f_{11} = \frac{L^3}{3EI}$$

$$f_{21} = -\frac{L^2}{2EI}$$

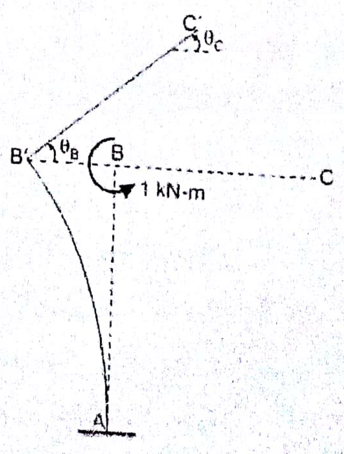
$$f_{31} = -\frac{L^2}{2EI}$$

Also, from Maxwell's reciprocal theorem,

$$f_{12} = f_{21} = \frac{-L^2}{2EI}$$

$$f_{13} = f_{31} = \frac{-L^2}{2EI}$$

Second column:



$$f_{12} = \frac{-L^2}{2EI} \text{ (Already known)}$$

$$f_{22} = \frac{L}{EI}$$

$$f_{32} = \frac{L}{EI}$$

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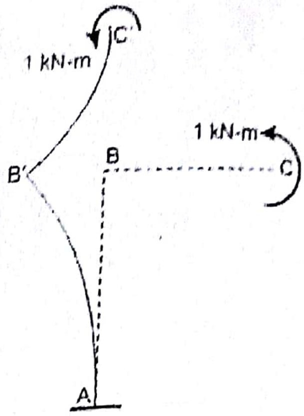
Also from Maxwell's reciprocal theorem,

$$f_{23} = f_{32} = \frac{L}{EI}$$



Third column:

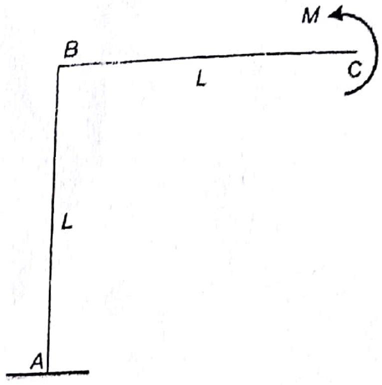
$$f_{13} = \frac{-L^2}{2EI} \text{ (Already known)}$$

$$f_{23} = \frac{L}{EI} \text{ (Already known)}$$

For f_{33},

Using strain-energy method

Strain-energy stored in frame



$$U = U_{AB} + U_{BC}$$

$$U = \frac{M^2L}{2EI} + \frac{M^2L}{2EI} = \frac{M^2L}{EI}$$

$$\theta_C = \frac{\partial U}{\partial M} = \frac{2ML}{EI}$$

If $M = 1$ kN-m, then, θ_c becomes

$$f_{33} = \frac{2L}{EI}$$

Hence the flexibility matrix for given cantilever frame is

$$[f] = \begin{bmatrix} \frac{L^3}{3EI} & \frac{-L^2}{2EI} & \frac{-L^2}{2EI} \\ -\frac{L^2}{2EI} & \frac{L}{EI} & \frac{L}{EI} \\ -\frac{L^2}{2EI} & \frac{L}{EI} & \frac{2L}{EI} \end{bmatrix}$$

10.7 Stiffness Matrix

10.7.1 Properties

1. The stiffness matrix is always square matrix having non-zero and non-negative diagonal elements.
2. The order of matrix = Degrees of freedom (D_k)

NOTE: (a) If load is vertical in beams, then axial displacement should be neglected
(b) If members are axially rigid, then also axial displacement should be ignored

10.7.2 Procedure to Develop Stiffness Matrix

If there are N coordinate then stiffness matrix will be $N \times N$ size square matrix. The element of stiffness matrix represents force produced by unit displacement in the direction of chosen coordinate

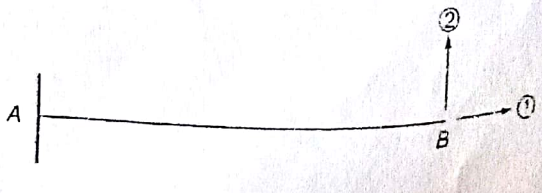


Fig. 10.5

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k_{xy} = Force/moment produced in the direction of x when unit displacement (Δ or θ) is applied in y-direction alone

It is also noticed that,

$k_{xy} = k_{yx}$ (According to Maxwell's reciprocal theorem)

Step-1. To generate first column of stiffness matrix, give unit displacement in the direction of coordinate (1) alone without any displacement in other coordinate directions (i.e. No Δ or θ at other coordinate) and measure forces developed in all coordinate directions.

Step-2. To generate second column of stiffness matrix, give unit displacement in the direction of coordinate (2) alone without any displacement in other coordinate directions (i.e. No Δ or θ at other coordinates) and measure forces developed in all coordinate directions.

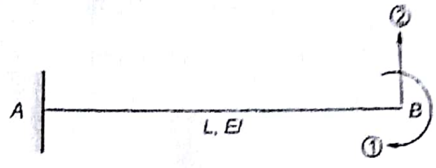


Fig. 10.6

Consider a cantilever beam with coordinate as shown in figure.

First column: To generate first column of stiffness matrix, give unit displacement in the direction of coordinate (1) alone without any displacement in the direction of other coordinate i.e. no Δ or θ at other coordinates and measure force produced in the directions of all coordinates.

$\therefore$ Give $\theta_B = 1$ and ensure $\Delta_B = 0$

So provide hinge support at B.

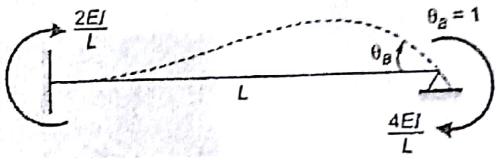


Fig. 10.7

k_{11} = Force developed in the direction of coordinate (1) when unit displacement is provided in the direction of coordinate (1).

k_{21} = Force developed in the direction of coordinate (2) when unit displacement is provided in the direction of coordinate (1).

Take,
$$\Sigma M_A = 0$$
$$R_B \times L - \frac{4EI}{L} - \frac{2EI}{L} = 0$$
$$R_B = \frac{6EI}{L^2}$$
$$\therefore \quad k_{11} = \frac{4EI}{L}$$
$$\therefore \quad k_{21} = R_B = \frac{6EI}{L^2}$$

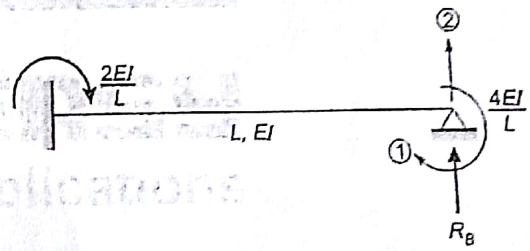


Fig. 10.8

Second column: To generate second column of stiffness matrix, give unit displacement in the direction coordinate (2) and measure force developed in the direction of coordinates (1) and (2).

$\therefore$ Give $\Delta_B = 1$ (↑) and ensure $\theta_B = 0$

So, fix the end B at B' so that

$$BB' = 1 \text{ unit}$$

Take,
$$\Sigma M_A = 0$$
$$\frac{6EI}{L^2} + \frac{6EI}{L^2} - R_B \times L = 0$$

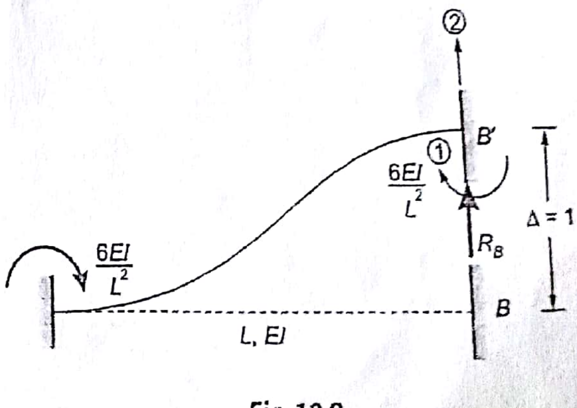


Fig. 10.9

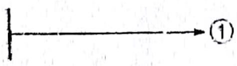
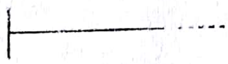
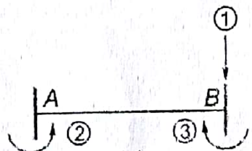
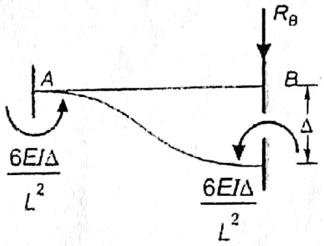
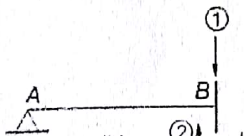
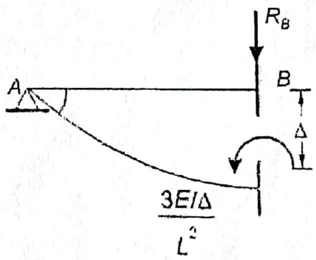
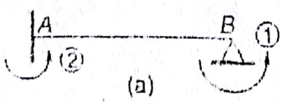
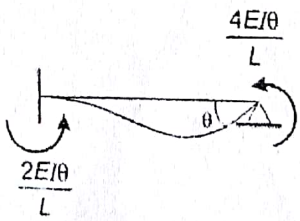
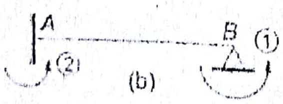
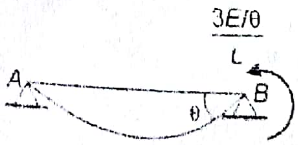
$$R_B = \frac{12EI}{L^3}(1)$$

$$k_{12} = \frac{6EI}{L^2} \quad \text{and} \quad k_{22} = R_B = \frac{12EI}{L^3}$$

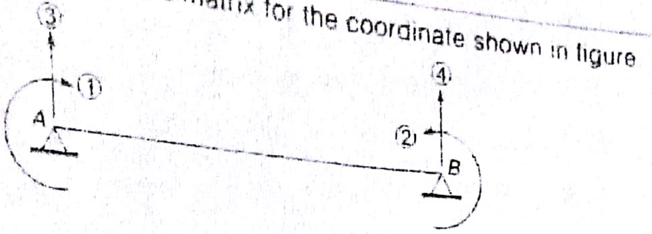
Hence the stiffness matrix for given beam is

$$[k] = \begin{bmatrix} \frac{4EI}{L} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{12EI}{L^3} \end{bmatrix}$$

10.7.3 Standard Results for Stiffness

Type of Displacement	Co-ordinate system	Displacement diagram	Stiffness
1. Axial			$k_{11} = \frac{AE}{L}$
2. Transverse displacement (a) with far end fixed	 (a)		$k_{11} = \frac{12EI}{L^3}$ $k_{21} = \frac{6EI}{L^2}$ $k_{31} = -\frac{6EI}{L^2}$
(b) with far end hinged	 (b)		$k_{11} = \frac{3EI\Delta}{L^3}$ $k_{21} = -\frac{3EI}{L^2}$
3. Flexural displacement (a) with far end fixed	 (a)		$k_{11} = \frac{4EI}{L}$ $k_{21} = \frac{2EI}{L}$
(b) with far end hinged	 (b)		$k_{11} = \frac{3EI}{L}$ $k_{21} = 0$

Example 10.9 Generate stiffness matrix for the coordinate shown in figure

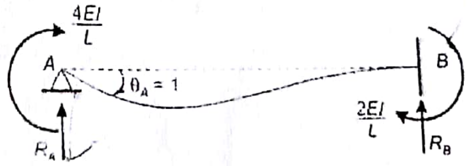


Solution:

First column. Give unit displacement in the direction of coordinate (1)

$\theta_A = 1$ and ensure $\Delta_A = 0$, $\Delta_B = 0$ and $\theta_B = 0$

So, replace support B by fixed support.



$\Sigma F_y = 0$

$$R_A + R_B = 0 \quad \ldots(i)$$

$\Sigma M_B = 0$,

$$R_A \times L + \frac{4EI}{L} + \frac{2EI}{L} = 0$$

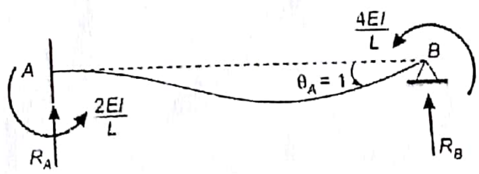
$$R_A = -\frac{6EI}{L^2} \text{ and } R_B = \frac{6EI}{L^2}$$

$$k_{11} = \frac{4EI}{L} \qquad k_{21} = -\frac{2EI}{L}$$

$$k_{31} = \frac{6EI}{L^2} \qquad k_{41} = \frac{6EI}{L^2}$$

Second column: Give unit displacement in the direction of coordinate (2)

$\theta_B = 1$ and ensure $\Delta_B = \Delta_A = 0$ and $\theta_A = 0$



$\Sigma F_y = 0$

$$R_A + R_B = 0$$

$\Sigma M_A = 0$

$$R_B \times L + \frac{4EI}{L} + \frac{2EI}{L} = 0$$

$$R_B = \frac{-6EI}{L^2} \text{ and } R_A = \frac{6EI}{L^2}$$

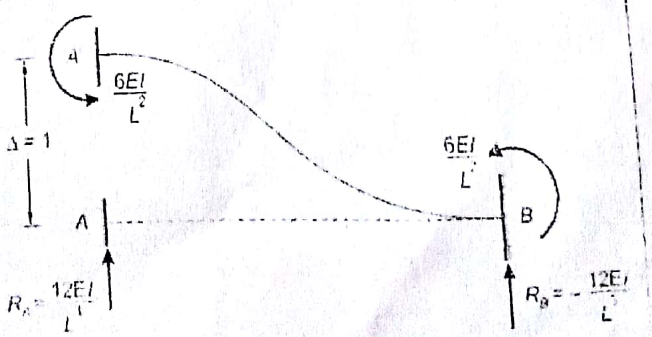
$$k_{12} = -\frac{2EI}{L} \qquad k_{22} = \frac{4EI}{L}$$

$$k_{32} = \frac{6EI}{L^2} \qquad k_{42} = \frac{-6EI}{L^2}$$

Third column: Give unit displacement in the direction of coordinate (3)

$\Delta = 1$ and ensure $\theta_A = \theta_B = 0$

$\Delta_B = 0$



$$K_{13} = \frac{-6EI}{L^2}$$

$$K_{23} = \frac{6EI}{L^2}$$

$$K_{33} = \frac{12EI}{L^3}$$

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$$k_{43} = \frac{-12EI}{L^3}$$

Fourth column: Give unit displacement in the direction of coordinate (4)

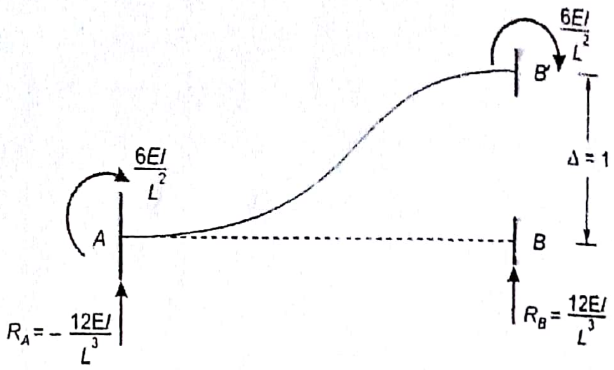
$\Delta_B = 1$ and ensure $\theta_A = \theta_B = 0$ and $\Delta_4 = 0$

$$k_{14} = \frac{6EI}{L^2}$$

$$k_{24} = -\frac{6EI}{L^2}$$

$$k_{34} = -\frac{12EI}{L^3}$$

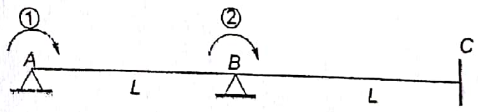
$$k_{44} = \frac{12EI}{L^3}$$



Hence the stiffness matrix for given coordinate system is

$$[k] = \begin{bmatrix} \frac{4EI}{L} & \frac{-2EI}{L} & \frac{-6EI}{L^2} & \frac{6EI}{L^2} \\ \frac{-2EI}{L} & \frac{4EI}{L} & \frac{6EI}{L^2} & \frac{-6EI}{L^2} \\ \frac{-6EI}{L^2} & \frac{6EI}{L^2} & \frac{12EI}{L^3} & \frac{-12EI}{L^3} \\ \frac{6EI}{L^2} & \frac{-6EI}{L^2} & \frac{-12EI}{L^3} & \frac{12EI}{L^3} \end{bmatrix}$$

Example 10.10 Generate stiffness matrix for coordinate shown in figure.



EI is constant

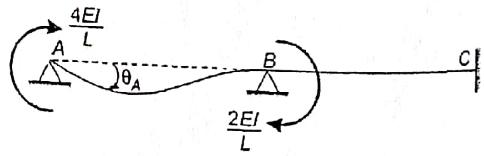
Solution:

First column: Give unit displacement in the direction of coordinate (1).

$\therefore$ $\theta_A = 1$ and ensure $\theta_B = 0$

$$k_{11} = \frac{4EI}{L}$$

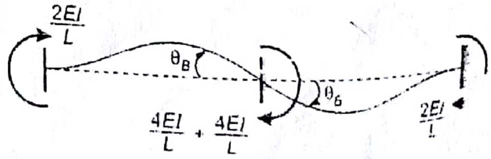
$$k_{21} = \frac{2EI}{L}$$



Second column: Give unit displacement in the direction of coordinate (2).

$\therefore$ $\theta_B = 1$ and ensure $\theta_A = \theta_C = 0$

$$k_{12} = \frac{2EI}{L}$$

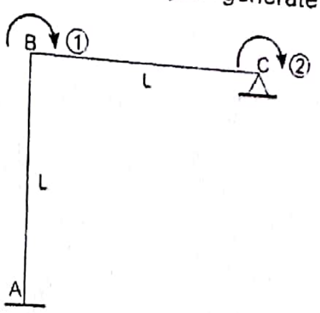


$$k_{22} = \frac{8EI}{L}$$

Hence the stiffness matrix for given beam is

$$[k] = \begin{bmatrix} \frac{4EI}{L} & \frac{2EI}{L} \\ \frac{2EI}{L} & \frac{8EI}{L} \end{bmatrix}$$

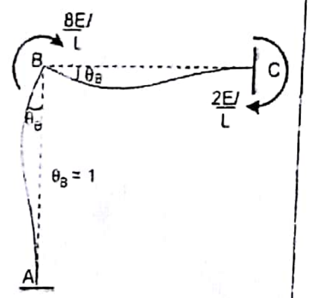
Example 10.11 For the frame shown in figure generate stiffness matrix



Solution:

First column: Give unit displacement in the direction of coordinate (1)

$\therefore$ $\theta_B = 1$ and ensure $\theta_C = 0$

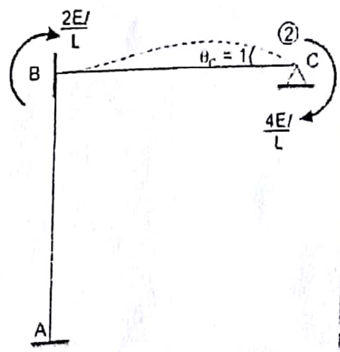


$$k_{11} = \frac{8EI}{L}$$

$$k_{21} = \frac{2EI}{L}$$

Second column: Give unit displacement in the direction of coordinate (2).

$\therefore$ $\theta_C = 1$ and ensure $\theta_B = 0$



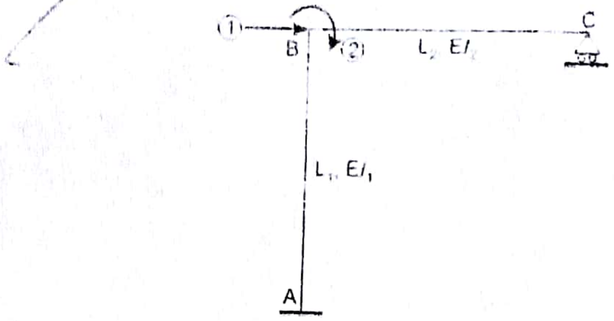
$$k_{12} = \frac{2EI}{L}$$

$$k_{22} = \frac{4EI}{L}$$

Hence stiffness matrix for given frame is

$$[k] = \begin{bmatrix} \frac{8EI}{L} & \frac{2EI}{L} \\ \frac{2EI}{L} & \frac{4EI}{L} \end{bmatrix}$$

Example 10.12 Draw stiffness matrix for the frame shown below.



Solution:

First column: Give unit displacement in the direction of coordinate (1).

∴ $\Delta_B = 1\ (\rightarrow)$ and ensure $\theta_B = 0$.

Take $$\Sigma M_B' = 0$$

$$H_A \times L_1 - \frac{6EI_1}{L_1^2} - \frac{6EI_1}{L_1^2} = 0$$

$$H_A = \frac{12EI_1}{L_1^3}$$

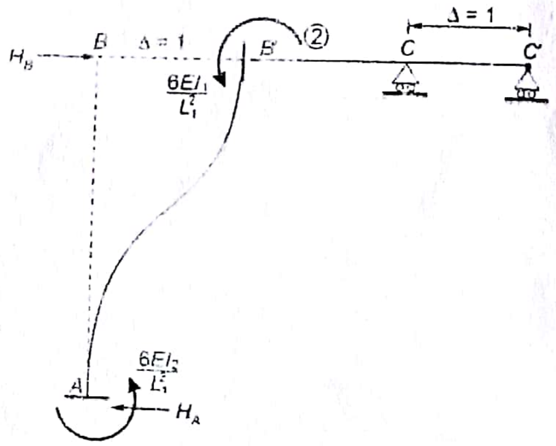
Also $$\Sigma F_x = 0$$

$$H_A = H_B$$

$$H_B = \frac{12EI_1}{L_1^3}$$

$$k_{11} = \frac{12EI_1}{L_1^3}$$

$$k_{21} = \frac{-6EI_1}{L_1^2}$$



Second column: Give unit displacement in the direction of coordinate (2).

∴ $\theta_B = 1$ and ensure $\Delta_B = 0\ (\rightarrow)$

$\Sigma F_x = 0$

$H_A = H_B$

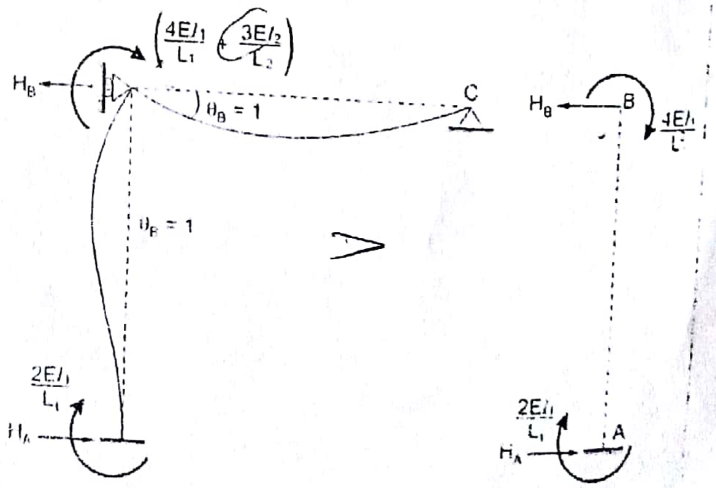
Also $\Sigma M_B = 0$

$$-H_A \times L_1 + \frac{2EI_1}{L_1} + \frac{4EI_1}{L_1} = 0$$

$$H_A = \frac{6EI_1}{L_1^2}$$

Hence $$H_B = \frac{6EI_1}{L_1^2}$$

$$k_{12} = -\frac{6EI_1}{L_1^2}$$



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$$k_{22} = \frac{4EI_1}{L_1} + \frac{3EI_2}{L_2}$$

Hence, stiffness matrix for given frame is

$$[k] = \begin{bmatrix} \frac{12EI_1}{L_1^3} & -\frac{6EI_1}{L_1^2} \\ \frac{-6EI_1}{L_1^2} & \left(\frac{4EI_1}{L_1} + \frac{3EI_2}{L_2}\right) \end{bmatrix}$$

10.8 Analysis of Beam and Frame Using Flexibility Matrix Method

10.8.1 Degree of Static Indeterminacy (D_S)

(a) **2D-Beam:** As the beam has open configuration, the degree of internal indeterminacy is zero. Hence the degree of static indeterminacy is given by,

$$D_S = r_e - 3$$

If beam has internal hinges, then the degree of static indeterminacy,

$$D_S = r_e - 3 - n$$

where, r_e = No. of independent external reactions

n = No. of internal hinges

(b) **2-D rigid frame:** For 2-D rigid frame, the degree of static indeterminacy is given by

$$D_S = 3m - r_e - 3j - r_r$$

where, m = number of members

r_e = number of independent external reactions

j = number of joints

r_r = number of reactions released

10.8.2 Basic Released Structure

It is statically determinate and stable structure which is obtained by releasing a sufficient number of internal forces or external reaction component in order to obtain determinate structure from corresponding statically indeterminate structure.

Consider a continuous beam *ABCD* as shown in figure.

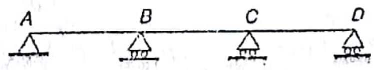

Fig 10.10

Degree of static indeterminacy for above beam is

$$D_S = r_e - 3$$

$$r_e = 5$$

Here,

$$D_S = 5 - 3 = 2$$

∴ Beam is statically indeterminate to second degree. Thus to

∴ make the beam statically determinate, two reactions component either internal or external have to be released.

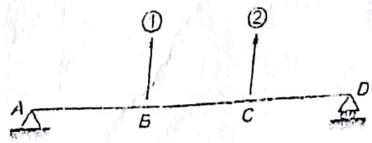

Fig 10.11 Released structure when R_B and R_C are released

Case-1: Consider external reactions at *B* and *C* are redundant. The released structure can be obtained by removing restraint offered by reactions. For above beam released structure is simply supported beam.

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Case-2: If bending moment at B and external support reaction at C is released. Then the released structure comprises two simply supported beam AB and BD as shown in figure

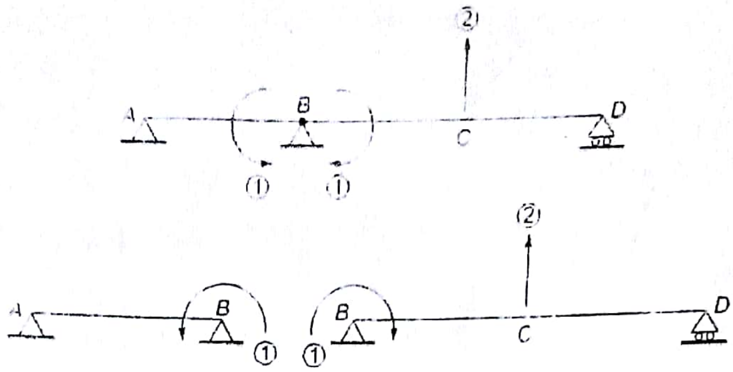

Fig. 10.12 Released structure when M_B and R_C are released

Case-3: If bending moment at B and C is released. Then the released structure comprises three simply supported beams AB, BC and CD as shown in figure

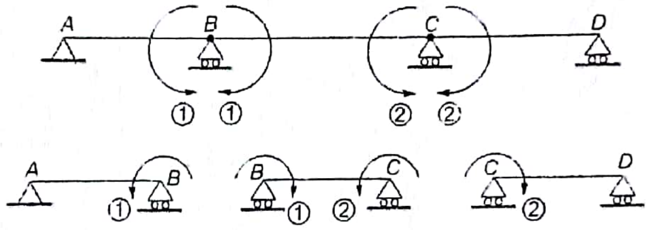

Fig. 10.13 Released structure when M_B and M_C are released

Procedure of Analysis

In flexibility method, unknown forces are taken as redundant and number of redundant are equal to degree of static indeterminacy. If there are N redundants, then flexibility matrix will be a square matrix of size $N \times N$

Consider a beam with coordinates as shown in figure.

f_{11} = Deflection in the direction of coordinate (1) when unit load is applied in the direction of coordinate (1).

f_{12} = Deflection in the direction of coordinate (1) when unit load is applied in the direction coordinate (2).

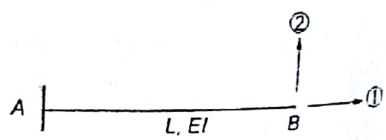

Fig. 10.14

If load P_1 is acting in the direction of coordinate (1) and load P_2 is acting in the direction of coordinate (2).

Thus the total deflection in the direction of coordinate (1) is,

$$\Delta_1 = f_{11}P_1 + f_{12}P_2 \qquad (i)$$

Similarly, the total deflection in the direction of coordinate (2) is,

$$\Delta_2 = f_{21}P_1 + f_{22}P_2 \qquad (ii)$$

The equations (i) and (ii) can be represented in matrix form as

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$$

$$[\Delta] = [f][P]$$
$$P = [f]^{-1}[\Delta]$$

Step-1. Find degree of static indeterminacy (D_S). Now identify the redundants in such a way that released structure remain stable and determinate. Neglect all axial effect in beams. Example

In above case there are two redundant say $R_A = R_1$ and $R_B = R_2$

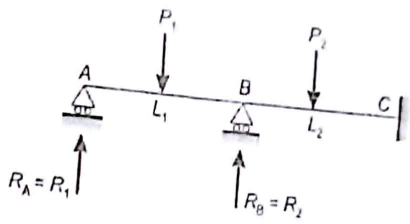

Fig. 10.15

Step-2. Remove the redundants and assign one coordinate in the direction of each redundant.

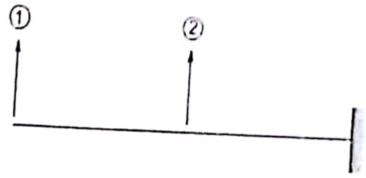

Fig. 10.16

Step-3. Develop flexibility matrix for above coordinate system and find inverse of flexibility matrix i.e. $[f]^{-1}$.

Step-4. Remove redundant and obtain basic released structure with given loading which is statically determinate and stable.

Step-5. For basic released structure, find deflection due to given loading in the direction of assign coordinate. Let Δ_{1L} and Δ_{2L} are deflections in the direction of coordinate (1) and (2) due to given loading in basic released structure.

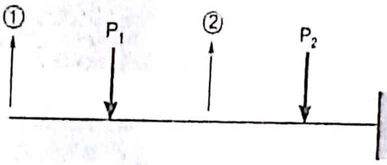

Fig. 10.17 *Basic released structure*

Step-6. Remove loading and apply redundant forces in the direction of assign coordinate and find deflection at coordinate (1) and (2). Let Δ_{1R} and Δ_{2R} are the displacements due to redundant reactions in the direction of coordinates (1) and (2) respectively.

$$\therefore \quad \Delta_{1R} = f_{11}R_1 + f_{12}R_2$$
$$\Delta_{2R} = f_{21}R_1 + f_{22}R_2$$

Hence,
$$\begin{bmatrix} \Delta_{1R} \\ \Delta_{2R} \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} \quad \ldots(i)$$

Find final deflection due to given loading and redundant reaction at coordinate (1) and (2).

$$\Delta_1 = \Delta_{1L} + \Delta_{1R}$$
$$\Delta_2 = \Delta_{2L} + \Delta_{2R}$$

Since in the direction of coordinate (1) and (2) redundant reactions are present. Hence final deflections will be zero.

$$\therefore \quad \Delta_{1R} = -\Delta_{1L}$$
$$\Delta_{2R} = -\Delta_{2L}$$

Also,
$$\begin{bmatrix} \Delta_{1R} \\ \Delta_{2R} \end{bmatrix} = \begin{bmatrix} -\Delta_{1L} \\ -\Delta_{2L} \end{bmatrix}$$

From equation (*i*), we get

$$\begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = \begin{bmatrix} -\Delta_{1L} \\ -\Delta_{2L} \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}^{-1} \begin{bmatrix} -\Delta_{1L} \\ -\Delta_{2L} \end{bmatrix} \quad \ldots(ii)$$

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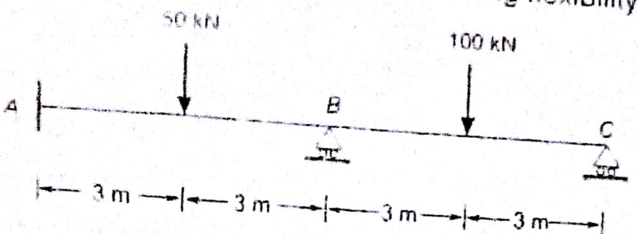
Example 10.13 Analyse the beam shown in figure using flexibility matrix method.



EI is constant

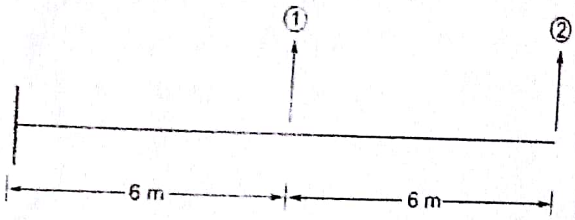
Solution:

$$D_S = r_e - 3$$

$$D_S = 5 - 3 = 2$$

Thus given continuous beam is redundant to second degree

Let us consider reactions R_B and R_C as redundant. Also let coordinate (1) in the direction of R_B and coordinate (2) in the direction of R_C



Co-ordinate System

Flexibility Matrix

Column 1st: Apply unit load in the direction of coordinate (1) and measure displacements in the direction of coordinate (1) and (2)

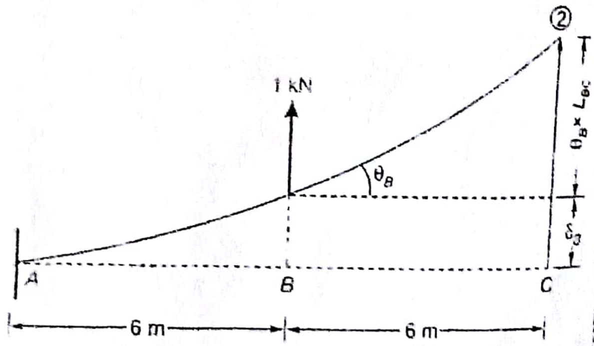
$$f_{11} = \frac{1 \times 6^3}{3EI} = \frac{72}{EI}$$

$$f_{21} = \delta_B + \theta_B \times L_{BC}$$

$$f_{21} = \frac{72}{EI} + \frac{1 \times (6)^2}{2EI} \times 6$$

$$= \frac{72}{EI} + \frac{108}{EI}$$

$$f_{21} = \frac{180}{EI}$$



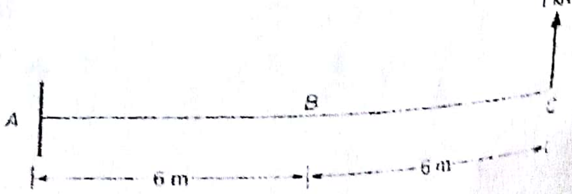
Also, from reciprocal theorem,

$$f_{12} = f_{21} = \frac{180}{EI}$$

Column 2nd: Apply unit load in the direction of coordinate (2) and measure displacement is the direction of coordinate (1) and (2)

$$f_{12} = \frac{180}{EI}$$

$$f_{22} = \frac{1 \times (12)^3}{3EI} = \frac{576}{EI}$$



Hence, the flexibility matrix for basic released structure is

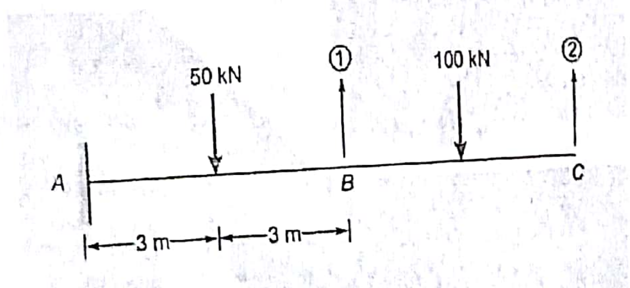
$$[f] = \begin{bmatrix} \dfrac{72}{EI} & \dfrac{180}{EI} \\ \dfrac{180}{EI} & \dfrac{576}{EI} \end{bmatrix} = \frac{1}{EI}\begin{bmatrix} 72 & 180 \\ 180 & 576 \end{bmatrix}$$

Inverse of flexibility matrix,

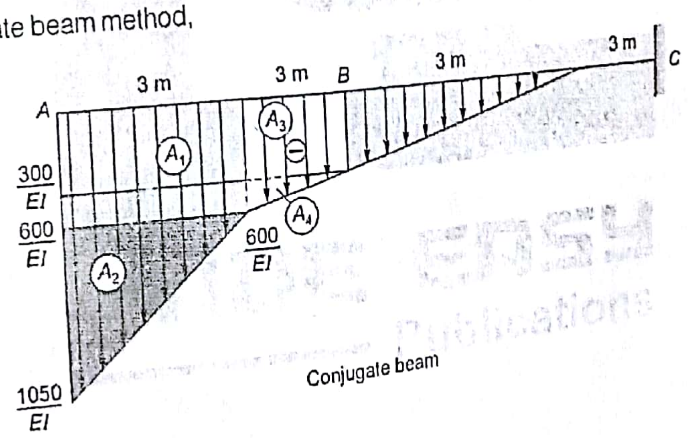
$$[f]^{-1} = \frac{Adj[f]}{|f|} = \frac{EI\begin{bmatrix} 576 & -180 \\ -180 & 72 \end{bmatrix}}{(72\times576-180\times180)}$$

$$\therefore \quad [f]^{-1} = EI\begin{bmatrix} 0.0634 & -0.0198 \\ -0.0198 & 0.0079 \end{bmatrix}$$

The displacements in the direction of coordinates due to external loading:



Using conjugate beam method,



Δ_{1L} = B.M at B

$$= A_1\bar{x}_1 + A_2\bar{x}_2 + A_3\bar{x}_3 + A_4\bar{x}_4$$

$$= -\frac{600}{EI}\times3\times4.5-\frac{1}{2}\times\frac{450}{EI}\times3\times\left(3+\frac{2}{3}\times3\right)-\frac{300}{EI}\times3\times1.5-\frac{1}{2}\times\frac{300}{EI}\times3\times\frac{2}{3}\times3$$

$$= -\frac{1}{EI}\left[600\times3\times4.5+\frac{450\times3\times5}{2}+300\times3\times1.5+300\times3\times1\right]$$

$$= -\frac{13725}{EI}\ (\downarrow)$$

BM at C

$$\Delta_{2L} = -\frac{600}{EI}\times 3\times 10.5-\frac{1}{2}\times\frac{450}{EI}\times 3\times\left(4+\frac{2}{3}\times 3\right)-\frac{1}{2}\times\frac{300}{EI}\times 3\times\left(6+\frac{2}{3}\times 3\right)$$

$$-\frac{300}{EI}\times 3\times(6+15)-\frac{1}{2}\times\frac{300}{EI}\times 3\left(3+\frac{2}{3}\times 3\right)$$

$$= -\frac{1}{EI}\left[600\times 3\times 10.5+\frac{450\times 3\times 11}{2}+\frac{300\times 3\times 8}{2}+300\times 3\times 7.5+\frac{300\times 3\times 5}{2}\right]$$

$$= -\frac{38925}{EI}(\downarrow)$$

$$\therefore \quad R = [f]^{-1}[-\Delta_L]$$

$$\begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = EI\begin{bmatrix} 0.0634 & -0.0198 \\ -0.0198 & 0.0079 \end{bmatrix}\times\frac{1}{EI}\begin{bmatrix} -(-13725) \\ -(-38925) \end{bmatrix}$$

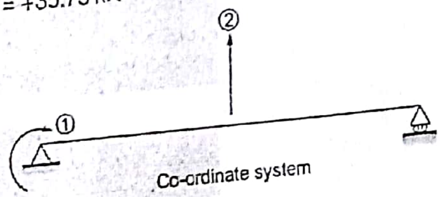
$$\begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = \begin{bmatrix} 0.0634 & -0.0198 \\ -0.0198 & 0.0079 \end{bmatrix}\begin{bmatrix} 13725 \\ 38925 \end{bmatrix}$$

$$\therefore \quad R_B = R_1 = 0.0634\times 13725-0.0198\times 38925 = +99.45\text{ kN}$$

$$R_C = R_2 = -0.0198\times 13725+0.0079\times 38925 = +35.75\text{ kN}$$

Alternate Solution

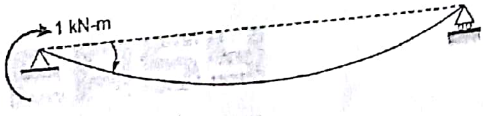
Now let us consider M_A and R_B as redundant. Also let coordinate (1) in the direction of M_A and coordinate (2) in the direction of R_B



Co-ordinate system

Flexibility Matrix

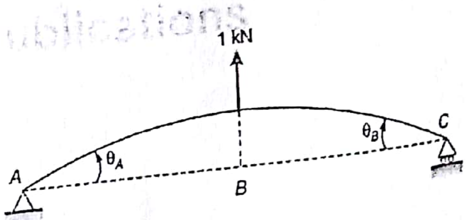
Column 1st:



$$f_{11} = \frac{1\times L}{3EI} = \frac{4}{EI}$$

$f_{21} = f_{12}$ (By Reciprocal theorem)

Column 2nd:



$$f_{12} = -\frac{1\times L^2}{16EI} = -\frac{9}{EI}$$

$$f_{22} = \frac{1\times L^3}{48EI} = \frac{36}{EI}$$

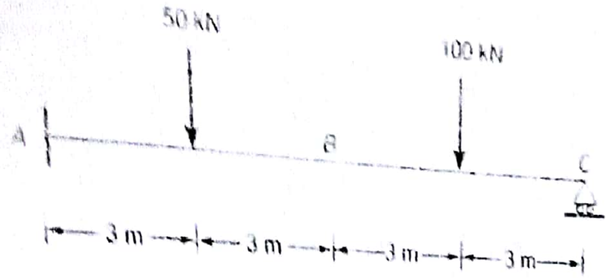
Hence flexibility matrix for selected coordinate system is

$$[f] = \frac{1}{EI}\begin{bmatrix} 4 & -9 \\ -9 & 36 \end{bmatrix}$$

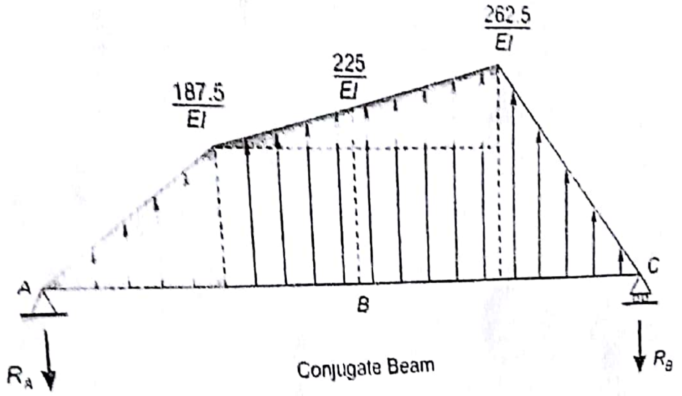
The inverse of flexibility matrix,

$$[f]^{-1} = \frac{Adj|f|}{|f|} = EI\begin{bmatrix} 0.5714 & +0.1428 \\ +0.1428 & 0.0634 \end{bmatrix}$$

The displacements in the direction of coordinates due to external loading



using conjugate beam method.



Conjugate Beam

$$R_A + R_B = \frac{1}{2}\times\frac{187.5}{EI}\times 3+\frac{187.5}{EI}\times 6+\frac{1}{2}\times\left(\frac{262.5}{EI}-\frac{187.5}{EI}\right)\times 6+\frac{1}{2}\times\frac{262.5}{EI}\times 3-\frac{2025}{EI} \qquad ...(i)$$

Applying $\Sigma M_C = 0$

$$\Rightarrow -R_A\times 12+\frac{1}{2}\times 3\times\frac{187.5}{EI}\times 10+\frac{187.5}{EI}\times 6\times 6+\frac{1}{2}\times 6\times\frac{75}{EI}\times 5+\frac{1}{2}\times\frac{262.5}{EI}\times 3\times\frac{2}{3}\times 3=0$$

$$\Rightarrow -12R_A+\frac{(2812.5+6750+1125+787.5)}{EI}=0$$

$$\Rightarrow \quad R_A = \frac{956.25}{EI} \qquad \left(R_B=\frac{2025}{EI}-R_A\right)$$

$$R_B = \frac{1068.75}{EI}$$

$$\therefore \quad \Delta_{1L} = \theta_A = \text{SF at A in C.B} = -\frac{956.25}{EI} \circlearrowright = +\frac{956.25}{EI} \quad \text{(in the direction of (2)}$$

$$\therefore \quad \Delta_{2L} = \text{B.M at } B \text{ in C.B}$$

$$= -R_A\times 6+\left(\frac{1}{2}\times 3\times\frac{187.5}{EI}\times 4\right)+\left(\frac{187.5}{EI}\times 3\times 1.5\right)+\left(\frac{1}{2}\times 3\times\frac{37.5}{EI}\times 1\right)$$

$$= -\frac{956.25\times 6}{EI}+\frac{3\times 187.5\times 4}{2EI}+\frac{187.5\times 3\times 1.5}{EI}+\frac{3\times 37.5}{2EI} = -\frac{7425}{EI} \;(\downarrow)$$

$$[R] = [f]^{-1}[-\Delta_L]$$

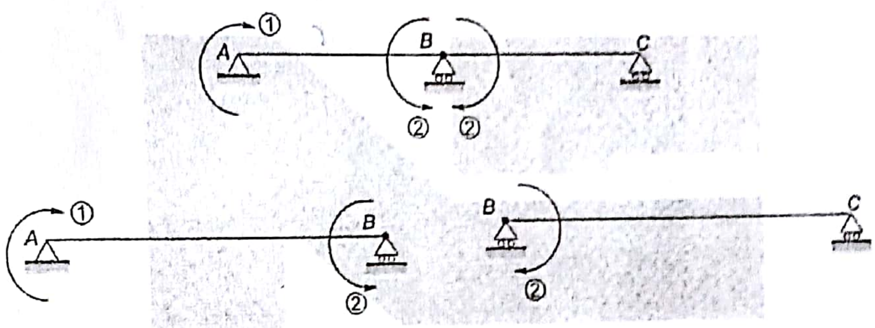
$$\begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = EI\begin{bmatrix} 0.5714 & 0.1428 \\ 0.1428 & 0.0634 \end{bmatrix} \times \frac{1}{EI}\begin{bmatrix} -956.25 \\ +3712.5 \end{bmatrix}$$

∴
$$R_1 = 0.5714 \times -956.25 + 0.1428 \times 3712.5$$
$$M_A = R_1 = -16.25 \text{ kN}$$

and
$$0.1428 \times -956.25 + 0.0634 \times 3712.5$$
$$R_B = R_2 = 98.82 \text{ kN} (\uparrow)$$

Alternate Solution

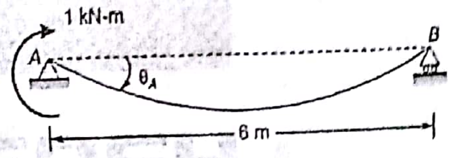
Let us consider M_A and M_B as redundant. Also let coordinate (1) in the direction of M_A and coordinate (2) in the direction of M_B.



Flexibility Matrix

Column 1st:

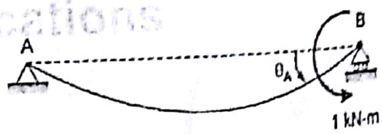
$$f_{11} = \frac{1 \times L}{3EI} = \frac{6}{3EI} = \frac{2}{EI}$$



$$f_{21} = \frac{L}{6EI} = \frac{1}{EI}$$

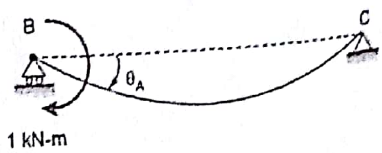
Column 2nd:

$$f_{12} = f_{21} = \frac{1}{EI}$$



$$f_{22} = (\theta_B)_{AB} + (\theta_B)_{BC}$$

$$f_{22} = \frac{L}{3EI} + \frac{L}{3EI} = \frac{2L}{3EI} = \frac{2 \times 6}{3EI}$$



$$f_{22} = \frac{4}{EI}$$

∴

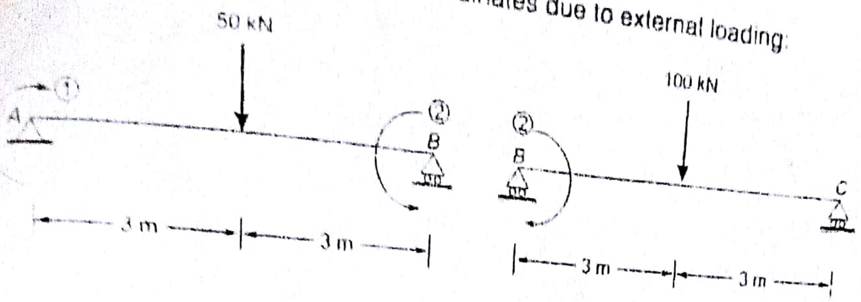
Hence flexibility matrix for selected coordinate is

$$[f] = \frac{1}{EI}\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$$

The inverse of flexibility matrix is

$$[f]^{-1} = \frac{Adj\,[f]}{|f|} = EI\begin{bmatrix} 0.5714 & -0.1428 \\ -0.1428 & 0.2857 \end{bmatrix}$$

The displacements in the direction of coordinates due to external loading:



$$\Delta_{1L} = \theta_A = \frac{50 \times 6^2}{16EI} = \frac{112.5}{EI}$$

$$\Delta_{2L} = \frac{50 \times 6^2}{16EI} + \frac{100 \times 6^2}{16EI} = \frac{337.5}{EI}$$

$$R = [f]^{-1}[-\Delta_l]$$

$$\begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = EI\begin{bmatrix} 0.5714 & -0.1428 \\ -0.1428 & 0.2857 \end{bmatrix} \times \frac{1}{EI}\begin{bmatrix} -112.5 \\ -337.5 \end{bmatrix}$$

$$M_A = R_1 = 0.5714 \times (-112.5) - 0.1428 \times (-337.5)$$

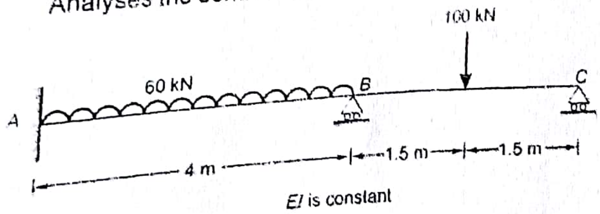
$$M_A = -16.08 \text{ kN-m}$$

and

$$M_B = R_1 = -0.1428 \times (-112.5) + 0.2857 \times (-337.5)$$

$$M_B = -80.35 \text{ kN-m}$$

Example 10.14 Analyses the continuous beam shown in figure. Use flexibility matrix method.



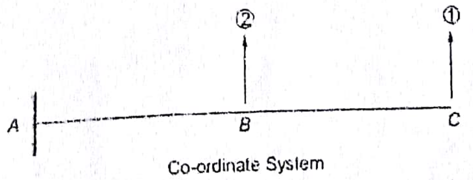
EI is constant

Solution:

$$D_S = 5 - 3 = 2$$

Thus the given beam is indeterminate to second degree.

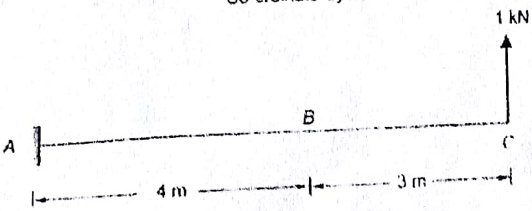
Let us consider R_C and R_B be the redundants. Also let coordinate (1) in the direction of R_C and coordinate (2) in the direction of R_B.



Co-ordinate System

Flexibility Matrix

Column 1st:



$$f_{11} = \frac{1 \times (7)^3}{3EI} = \frac{343}{3EI}$$

$$f_{21} = f_{12} \quad \text{(From Maxwell's reciprocal theorem)}$$

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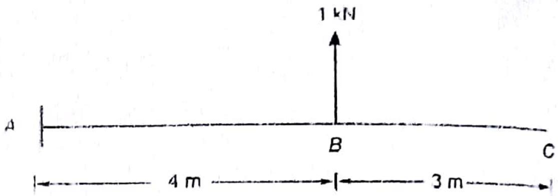
Column 2nd



$$f_{12} = \frac{1\times 4^3}{3EI} + \frac{1\times 4^2}{2EI}\times 3$$

$$f_{12} = \frac{136}{3EI}$$

$$f_{22} = \frac{1\times 4^3}{3EI} = \frac{64}{3EI}$$

Hence the flexibility matrix for selected coordinate is

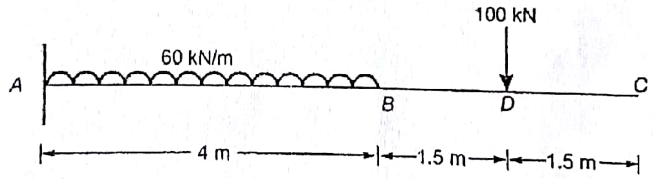
$$[f] = \frac{1}{EI}\begin{bmatrix} \frac{343}{3} & \frac{136}{3} \\ \frac{136}{3} & \frac{64}{3} \end{bmatrix}$$

The inverse of flexibility matrix is

$$[f]^{-1} = \frac{Adj[f]}{|f|} = \frac{EI}{|f|}\begin{bmatrix} \frac{64}{3} & \frac{-136}{3} \\ \frac{-136}{3} & \frac{343}{3} \end{bmatrix}$$

$$\therefore \quad [f]^{-1} = EI\begin{bmatrix} 0.055 & -0.118 \\ -0.118 & 0.297 \end{bmatrix}$$

This displacement in the direction of coordinates due to external loading:



$$\Delta_{1L} = \Delta_C = -\left[\frac{100\times(AD)^3}{3EI} + \frac{100\times(AD)^2}{2EI}\times CD + \frac{60\times(AB)^4}{8EI} + \frac{60\times(AB)^3}{6EI}\times BC\right]$$

$$= -\left[\frac{100\times 5.5^3}{3EI} + \frac{100\times 5.5^2}{2EI}\times 1.5 + \frac{60\times 4^4}{8EI} + \frac{60\times 4^3}{6EI}\times 3\right] = -\frac{11654.58}{EI}$$

$$\Delta_{2L} = \Delta_B = -\left[\frac{100\times(AB)^3}{3EI} + \frac{100\times(AB)^2}{2EI}\times BD + \frac{60\times(AB)^4}{8EI}\right]$$

$$\Delta_{2L} = -\frac{5253.33}{EI}$$

$$[R] = [f]^{-1}[-\Delta_L]$$

$$\begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = EI\begin{bmatrix} 0.055 & -0.118 \\ -0.118 & 0.297 \end{bmatrix}\times\frac{1}{EI}\begin{bmatrix} -(-11654.58) \\ -(-5253.33) \end{bmatrix}$$

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$$R_C = R_1 = 0.055 \times 11654.58 - 0.118 \times 5253.33$$
$$R_C = 21.10 \text{ kN}$$

and

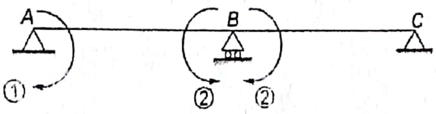
$$R_B = R_2 = -0.118 \times 11654.58 + 0.297 \times 5253.33$$
$$R_B = 185 \text{ kN}$$

$$\Sigma F_y = 0$$

$$R_A + R_B + R_C = 100 + 60 \times 4$$
$$R_A = 340 - 21.10 - 185 = 133.9 \text{ kN}$$

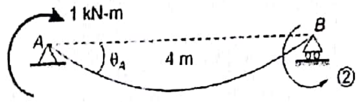
Alternate Solution

Now let us select M_A and M_B as redundant.



Flexibility Matrix

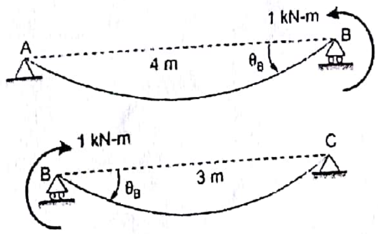
Column 1st:



$$f_{11} = \frac{4}{3EI}$$

$$f_{21} = \frac{4}{6EI}$$

Column 2nd:



$$f_{12} = \frac{4}{6EI}$$

$$f_{22} = (\theta_B)_{AB} + (\theta_B)_{BC}$$

$$= \frac{4}{3EI} + \frac{3}{3EI} = \frac{7}{3EI}$$

Hence the flexibility matrix for selected coordinate is

$$[f] = \frac{1}{EI}\begin{bmatrix} \frac{4}{3} & \frac{4}{6} \\ \frac{4}{6} & \frac{7}{3} \end{bmatrix}$$

The inverse of flexibility matrix is

$$[f]^{-1} = \frac{Adj\,[f]}{|f|} = \frac{EI}{2.654}\begin{bmatrix} 2.33 & -0.667 \\ -0.667 & 1.33 \end{bmatrix}$$

$$= EI\begin{bmatrix} 0.877 & -0.251 \\ -0.251 & 0.501 \end{bmatrix}$$

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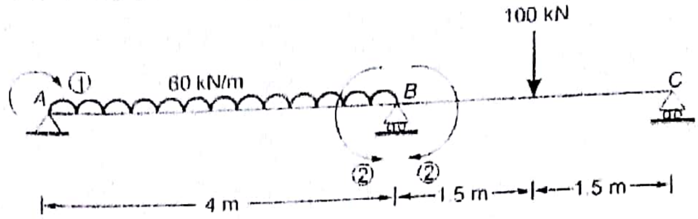
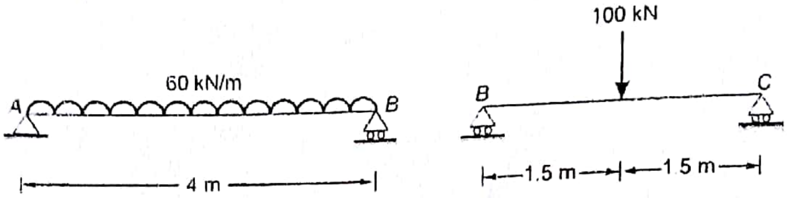
The displacements in the direction of coordinates due to external loading:





$$\Delta_{1L} = \theta_A = \frac{60 \times 4^3}{24EI} = \frac{160}{EI}$$

$$\Delta_{2L} = (\theta_B)_{AB} + (\theta_B)_{BC}$$

$$= \frac{60 \times 4^3}{24EI} + \frac{100 \times 3^2}{16EI} = \frac{216.25}{EI}$$

$$[R] = [f]^{-1}[-\Delta_L]$$

$$\begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = EI\begin{bmatrix} 0.877 & -0.251 \\ -0.251 & 0.501 \end{bmatrix} \times \frac{1}{EI}\begin{bmatrix} -160 \\ -216.25 \end{bmatrix}$$

$$R_1 = 0.877 \times (-160) + (-0.251) \times (-216.25)$$

$$M_A = R_1 = -86.04 \text{ kN-m}$$

$$M_B = R_2 = -0.251 \times (-160) + 0.501 \times (-216.25) = -68.18 \text{ kN-m}$$

10.9 Analysis of Beam and Frame Using Stiffness Matrix Method

Procedure of Analysis using Stiffness Matrix Method

Step-1. Determine degree of kinematic indeterminacy neglecting axial deformations. Identify independent joint displacement $\Delta_1, \Delta_2, \ldots\ldots \Delta_n$.

Step-2. Assign one coordinate in the direction of each unknown displacement of joint. Develop stiffness matrix and find inverse of matrix.

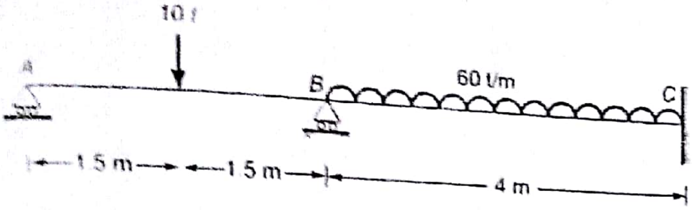
Step-3. Remove displacements in the direction of coordinates and obtain locked structure. Find the forces developed in the direction of assign coordinate due to given loading in the locked structure.

Let $P_{1L}, P_{2L} \ldots P_{nL}$ are forces developed in the direction of coordinate. Then displacement $\Delta_1, \Delta_2, \Delta_3 \ldots \Delta_n$ will be

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \vdots \\ \Delta_n \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} \ldots\ldots k_{1n} \\ k_{21} & k_{22} \ldots\ldots k_{2n} \\ \vdots & \vdots \\ k_{n1} & \ldots\ldots k_{nn} \end{bmatrix} \begin{bmatrix} -P_{1L} \\ -P_{2L} \\ -P_{3L} \end{bmatrix}$$

$$\Delta = [k]^{-1}[-P_L]$$

Example 10.15 Using stiffness matrix method analyses the beam shown in figure. Find final ... and also draw bending moment diagram.



EI is constant

Solution:

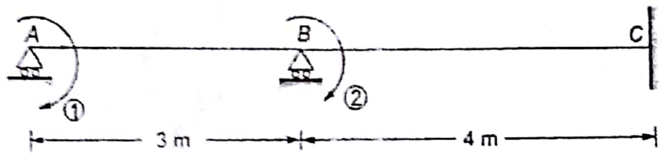
$$D_K = 3j - r_e - m''$$

Here $j = 3$, $r_e = 5$ and m'' = axially rigid member = 2

$$D_K = 3 \times 3 - 5 - 2$$

$$D_K = 2$$

Take θ_A and θ_B as unknown displacement. ... coordinate (1) in the direction of θ_A and ... (2) in the direction θ_B

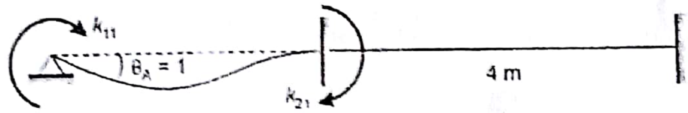


Stiffness Matrix

Column 1st:

Give unit displacement in the direction of coordinate (1).

$\theta_A = 1$ and ensure $\theta_B = 0$



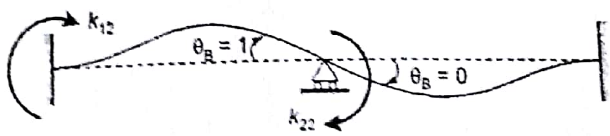
$$k_{11} = \frac{4EI}{3}$$

$$k_{21} = \frac{2EI}{3}$$

Column 2nd:

Give unit displacement in the direction of coordinate (2).

$\theta_B = 1$ and ensure $\theta_A = 0$



$$k_{12} = \frac{2EI}{3}$$

$$k_{22} = \frac{4EI}{3} + \frac{4EI}{4} = \frac{7}{3}EI$$

The stiffness matrix for selected coordinate is

$$[K] = \begin{bmatrix} \frac{4EI}{3} & \frac{2EI}{3} \\ \frac{2EI}{3} & \frac{7EI}{3} \end{bmatrix} = \frac{EI}{3}\begin{bmatrix} 4 & 2 \\ 2 & 7 \end{bmatrix}$$

The inverse of stiffness matrix is

$$[K]^{-1} = \frac{1}{8EI}\begin{bmatrix} 7 & -2 \\ -2 & 4 \end{bmatrix}$$

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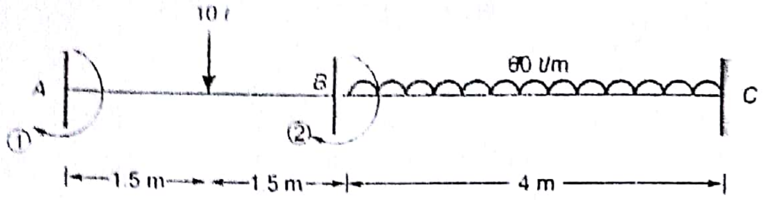
Let P_{1L} and P_{2L} are forces (Moments) developed in the coordinate directions due to the given load in locked structure.



$$\overline{M}_{AB} = -\frac{10\times 3}{8} = -3.75 \text{ t-m}$$

$$\overline{M}_{BA} = 3.75 \text{ t-m}$$

$$\overline{M}_{BC} = -\frac{6\times 4^2}{12} = -8 \text{ t-m}$$

$$\overline{M}_{CB} = 8 \text{ t-m}$$

$$P_{1L} = \overline{M}_{AB} = -3.75 \text{ t-m}$$

$$P_{2L} = \overline{M}_{AB} - \overline{M}_{BC}$$
$$= 3.75 - 8 = -4.25 \text{ t-m}$$

$$\begin{bmatrix}\Delta_1\\ \Delta_2\end{bmatrix} = \frac{1}{8EI}\begin{bmatrix}7 & -2\\ -2 & 4\end{bmatrix}\begin{bmatrix}-(-3.75)\\ -(-4.25)\end{bmatrix}$$

$$\Delta_1 = \frac{1}{8EI}[7\times 3.75 - 2\times 4.25]$$

$\therefore$
$$\Delta_1 = \theta_A = \frac{2.22}{EI}$$

and
$$\Delta_2 = \frac{1}{8EI}[-2\times 3.75 + 4\times 4.25]$$

$\therefore$
$$\Delta_2 = \theta_B = \frac{1.1875}{EI}$$

Final end moments:

$$M_{AB} = \overline{M}_{AB} + \frac{2EI}{L}\left(2\theta_A + \theta_B - \frac{3\Delta}{L}\right)$$

$$M_{AB} = -3.75 + \frac{2EI}{3}\left(2\times\frac{2.22}{EI} + \frac{1.1875}{EI} - 0\right)$$

$$M_{AB} = 0 \text{ t-m}$$

$$M_{BA} = \overline{M}_{BA} + \frac{2EI}{L}\left(2\theta_B + \theta_A - \frac{3\Delta}{L}\right)$$

$$= 3.75 + \frac{2EI}{3}\left(\frac{2\times 1.1875}{EI} + \frac{2.22}{EI} - 0\right) = 6.81 \text{ t-m}$$

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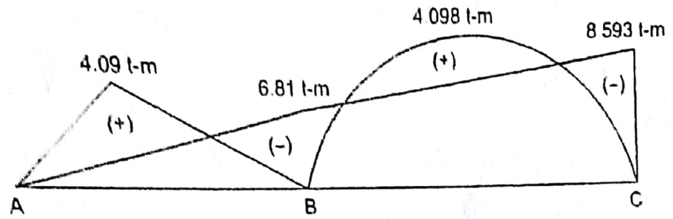
$$M_{BC} = \overline{M}_{BC} + \frac{2EI}{L}\left(2\theta_B + \theta_C - \frac{3\Delta}{L}\right)$$

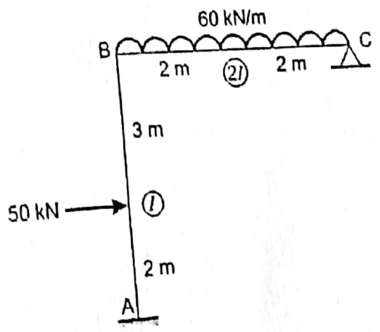
$$= -8 + \frac{2EI}{4}\left(\frac{2 \times 1.1875}{EI} + 0 - 0\right) = -6.81 \text{ t-m}$$

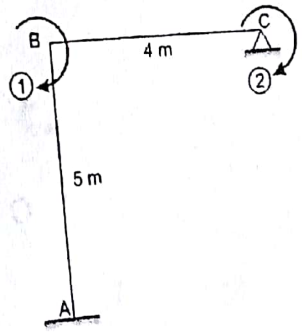
$$M_{CB} = \overline{M}_{CB} + \frac{2EI}{L}\left(\theta_B + 2\theta_C - \frac{3\Delta}{L}\right)$$

$$= 8 + \frac{2EI}{4}\left(\frac{1.1875}{EI} + 0 - 0\right) = 8.593 \text{ t-m}$$



Example 10.16 Analysis the frame shown in figure using stiffness matrix method.





Solution:

$$D_K = 3j - r_e - m''$$

Here, $j = 3$, $r_e = 5$, $m'' = 2$

$$D_K = 3 \times 3 - 5 - 2 = 2$$

∴

Take θ_B and θ_C as unknown displacement. Assign coordinate (1) in the direction of θ_A and coordinate (2) in the direction of θ_B.

Stiffness Matrix

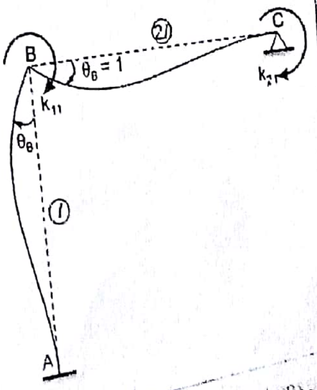
Column 1st:

Give unit displacement in the direction of coordinate (1).

∴ $\theta_B = 1$ and ensure $\theta_C = 0$

$$k_{11} = \frac{4EI}{5} + \frac{4E(2I)}{4} = \frac{14EI}{5}$$

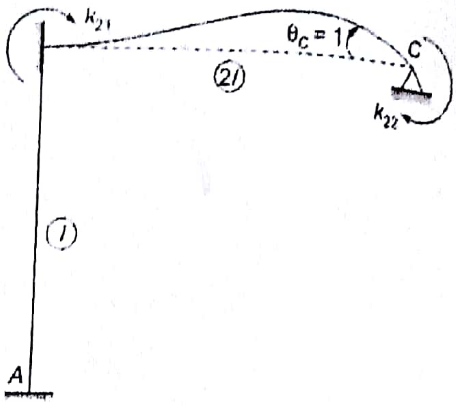
$$k_{21} = \frac{2E(2I)}{4} = EI$$



Column 2nd.

Give unit displacement in the direction of coordinate (2)

$\theta_C = 1$ and ensure $\theta_B = 0$



$$k_{12} = k_{21} = EI$$

$$k_{22} = \frac{4E(2I)}{4} = 2EI$$

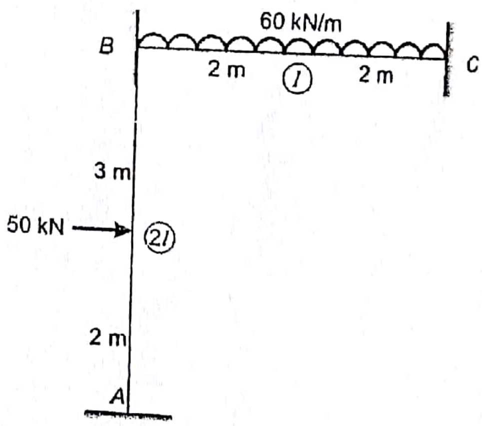
Hence, the stiffness matrix for selected coordinate is

$$[k] = \begin{bmatrix} \frac{14}{5}EI & EI \\ EI & 2EI \end{bmatrix}$$

The inverse of stiffness matrix is

$$[k]^{-1} = \frac{1}{23EI}\begin{bmatrix} 10 & -5 \\ -5 & 14 \end{bmatrix} \text{ or } \frac{1}{EI}\begin{bmatrix} 0.434 & -0.217 \\ -0.217 & 0.608 \end{bmatrix}$$

Let P_{1L} and P_{2L} are forces (moments) developed in the coordinate directions due to the given loading in locked structure.



$$\overline{M}_{AB} = -\frac{Pab^2}{L^2} = -\frac{50 \times 2 \times 3^2}{5^2} = -36 \text{ kN-m}$$

$$\overline{M}_{BA} = \frac{Pba^2}{L^2} = \frac{50 \times 3 \times 2^2}{5^2} = 24 \text{ kN-m}$$

$$\overline{M}_{BC} = -\frac{wl^2}{12} = -\frac{60 \times 4^2}{12} = -80 \text{ kN-m}$$

$$\overline{M}_{CB} = \frac{wl^2}{12} = \frac{60 \times 4^2}{12} = 80 \text{ kN-m}$$

$\therefore$ $$P_{1L} = \overline{M}_{BA} - \overline{M}_{BC}$$

$\Rightarrow$ $$P_{1L} = 24 - 80 = -56 \text{ kN-m}$$

and $$P_{2L} = \overline{M}_{CB}$$

$\therefore$ $$P_{2L} = 80 \text{ kN}$$

Hence $$\begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} = [k]^{-1}\begin{bmatrix} -P_{1L} \\ -P_{2L} \end{bmatrix}$$

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} = \frac{1}{EI}\begin{bmatrix} 0.434 & -0.217 \\ -0.217 & 0.608 \end{bmatrix}\begin{bmatrix} -(-56) \\ -(80) \end{bmatrix}$$

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} = \frac{1}{EI}\begin{bmatrix} 0.434 & -0.217 \\ -0.217 & 0.608 \end{bmatrix}\begin{bmatrix} 56 \\ -80 \end{bmatrix}$$

$$\Delta_1 = \frac{1}{EI}[0.434 \times 56 + (-0.217) \times (-80)]$$

$$\theta_B = \Delta_1 = \frac{41.664}{EI}$$

$$\Delta_2 = \frac{1}{EI}[-0.217 \times 56 + 0.608 \times (-80)]$$

$$\theta_C = \Delta_2 = -\frac{60.792}{EI}$$

Final and moments:

$$M_{AB} = \overline{M}_{AB} + \frac{2EI}{L_{AB}}\left(2\theta_A + \theta_B - \frac{3\Delta}{L}\right)$$

$$= -36 + \frac{2EI}{5}\left(0 + \frac{41.664}{EI} - 0\right) = -19.33 \text{ kN-m}$$

$$M_{BA} = \overline{M}_{BA} + \frac{2EI}{L_{AB}}\left(\theta_A + 2\theta_B - \frac{3\Delta}{L}\right)$$

$$= 24 + \frac{2EI}{5}\left(0 + \frac{2 \times 41.664}{EI} - 0\right) = 57.46 \text{ kN-m}$$

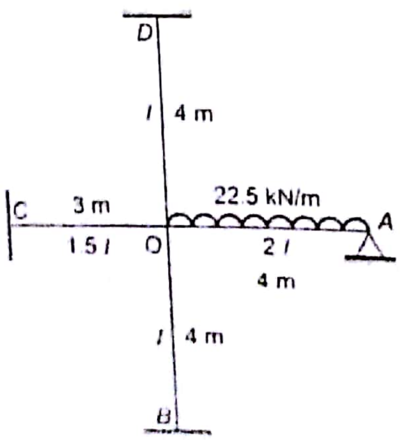
$$M_{BC} = \overline{M}_{BC} + \frac{2E(2I)}{L_{BC}}\left(2\theta_B + \theta_C - \frac{3\Delta}{L}\right)$$

$$= -80 + \frac{4EI}{4}\left(2 \times \frac{41.664}{EI} - \frac{60.792}{EI}\right) = -57.46 \text{ kN-m}$$

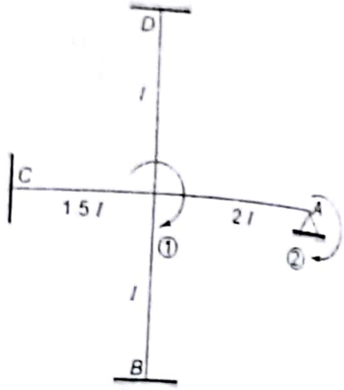
$$M_{CB} = \overline{M}_{CB} + \frac{2E(2I)}{L_{BC}}\left(\theta_B + 2\theta_C - \frac{3\Delta}{L}\right)$$

$$M_{CB} = +80 + \frac{4EI}{4}\left(\frac{41.664}{EI} - \frac{2 \times 60.792}{EI} - 0\right) = 0$$

Example 10.17 Analyse the frame shown in figure using stiffness matrix method.



Solution:



$$D_K = 3j - r_e - m''$$

Here $j = 5$, $r_e = 11$

$m'' = 2$ (Axial displacement of OD and OB already prevented i.e fixed at both end)

$$D_K = 3 \times 5 - 11 - 2 = 2$$

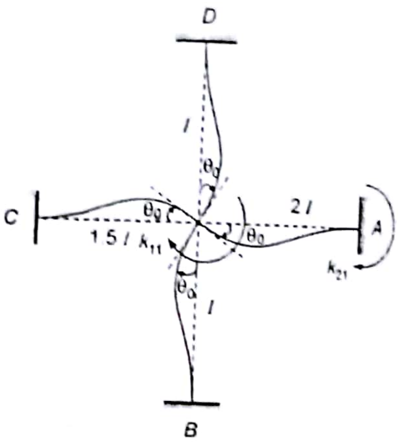
Take θ_0 and θ_A as unknown displacement. Assign coordinate (1) in the direction of θ_0 and coordinate (2) in the direction of θ_A

Stiffness Matrix

Column 1st

Give unit displacement in the direction of coordinate (1)

$\theta_0 = 1$ and ensure $\theta_A = 0$



$$k_{11} = \frac{4E(2I)}{4} + \frac{4EI}{4} + \frac{4E(1.5I)}{3} + \frac{4EI}{4}$$

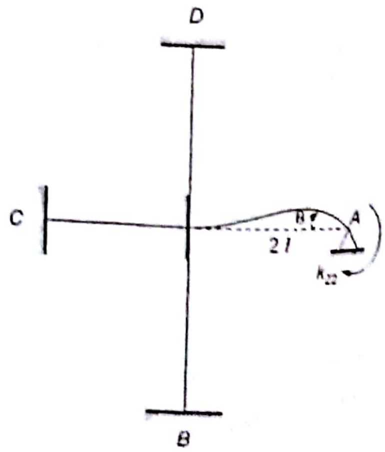
$$k_{11} = 6EI$$

$$k_{21} = \frac{2E(2I)}{4} = EI$$

Column 2nd

Give unit displacement in the direction of coordinate (2).

$\theta_A = 1$ and ensure $\theta_0 = 0$



$$k_{12} = k_{21} = EI$$

$$k_{22} = \frac{4E(2I)}{4} = 2EI$$

The stiffness matrix for selected coordinate is,

$$[k] = \begin{bmatrix} 6EI & EI \\ EI & 2EI \end{bmatrix}$$

The inverse of stiffness matrix is

$$[k]^{-1} = \frac{Adj\,[k]}{|k|} = \frac{1}{11EI}\begin{bmatrix} 2 & -1 \\ -1 & 6 \end{bmatrix}$$

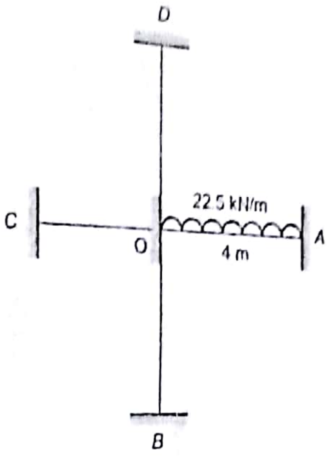
Let P_{1L} and P_{2L} are forces (moments) developed in the coordinate directions due to the given loading in locked structure

$$P_{1L} = \bar{M}_{OA} = -\frac{22.5 \times 4^2}{12}$$

$$P_{1L} = -30 \text{ kN-m}$$

$$P_{2L} = \bar{M}_{AO} = +30 \text{ kN-m}$$

Hence, $\begin{bmatrix}\Delta_1\\ \Delta_2\end{bmatrix} = \dfrac{1}{11EI}\begin{bmatrix}2 & -1\\ -1 & 6\end{bmatrix}\begin{bmatrix}-(-30)\\ -(30)\end{bmatrix}$



$$\therefore \quad \Delta_1 = \frac{1}{11EI}(2\times 30-1\times(-30))$$

$$\Rightarrow \quad \theta_0 = \Delta_1 = \frac{8.18}{EI}$$

and $\quad \Delta_2 = -1\times 30+6\times -30$

$$\Rightarrow \quad \theta_A = \Delta_2 = \frac{-19.09}{EI}$$

Final end moments:

$$M_{AO} = \overline{M}_{AO} + \frac{2E(2I)}{L}\left(2\theta_A+\theta_0-\frac{3\Delta}{L}\right)$$

$$= 30+\frac{4EI}{4}\left(\frac{2\times -19.09}{EI}+\frac{8.18}{EI}-0\right) = 0$$

$$M_{OA} = \overline{M}_{OA} + \frac{2E(2I)}{L}\left(\theta_A+2\theta_0-\frac{3\Delta}{L}\right)$$

$$= -30+\frac{4EI}{4}\left(-\frac{19.09}{EI}+\frac{2\times 8.18}{EI}-0\right) = -32.73 \text{ kN-m}$$

$$M_{BO} = \overline{M}_{BO} + \frac{2EI}{4}\left(2\theta_B+\theta_0-\frac{3\Delta}{L}\right)$$

$$= 0+\frac{2EI}{4}\left(0+\frac{8.18}{EI}\right) = 4.09 \text{ kN-m}$$

$$M_{OB} = \overline{M}_{OB} + \frac{2EI}{4}\left(\theta_B+2\theta_0-\frac{3\Delta}{L}\right)$$

$$= 0+\frac{2EI}{4}\left(0+\frac{2\times 8.18}{EI}-0\right) = 8.18 \text{ kN-m}$$

$$M_{CO} = \overline{M}_{CO} + \frac{2E(1.5EI)}{4}\left(2\theta_C+\theta_0-\frac{3\Delta}{L}\right)$$

$$= 0+\frac{3EI}{4}\left(0+\frac{8.18}{EI}-0\right) = 6.135 \text{ kN-m}$$

$$M_{OC} = \overline{M}_{OC} + \frac{2E(1.5I)}{4}\left(\theta_C+2\theta_0-\frac{3\Delta}{L}\right)$$

$$= 0+\frac{3EI}{4}\left(0+\frac{2\times 8.18}{EI}-0\right) = 12.27 \text{ kN-m}$$

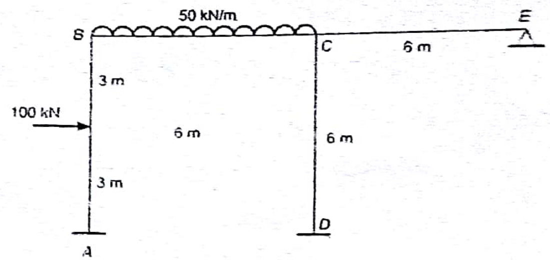
$$M_{DC} = \bar{M}_{DC} + \frac{2EI}{4}\left(2\theta_D + \theta_C - \frac{3\Delta}{L}\right)$$

$$= 0 + \frac{2EI}{4}\left(0 + \frac{8.18}{EI} - 0\right) = 4.09 \text{ kN-m}$$

$$M_{CD} = \bar{M}_{CD} + \frac{2EI}{4}\left(\theta_D + 2\theta_C - \frac{3\Delta}{L}\right)$$

$$= 0 + \frac{2EI}{4}\left(0 + \frac{2 \times 8.18}{EI} - 0\right) = 8.18 \text{ kN-m}$$

Example 10.18 Using stiffness matrix method, analyse the frame shown in figure.



EI is constant

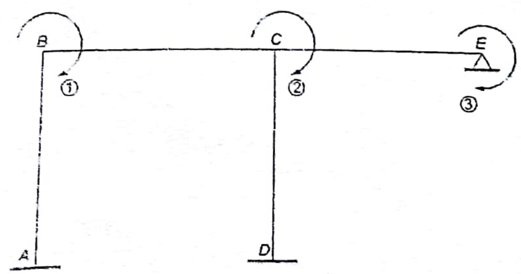
Solution:

$$D_K = 3j - r_e - m''$$

Here, $j = 5$, $r_e = 8$, $m'' = 4$

$$D_K = 3 \times 5 - 8 - 4$$
$$= 3$$

Take θ_B, θ_C and θ_E as unknown displacement. Assign coordinate (1) in the direction of θ_B, coordinate (2) in the direction of θ_C and coordinate (3) in the direction of θ_E.



Column 1st:

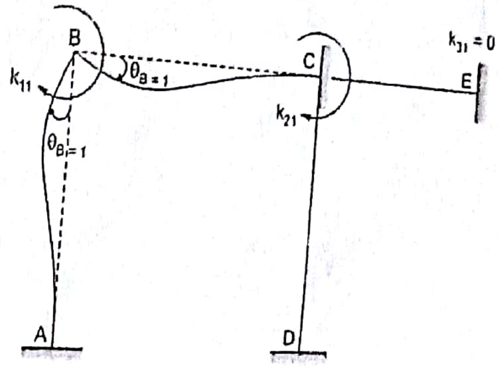
Give unit displacement in the direction of coordinate (1).

$\theta_B = 1$ and ensure $\theta_C = \theta_E = 0$

$$k_{11} = \frac{4EI}{6} + \frac{4EI}{6} = \frac{4}{3}EI$$

$$k_{21} = \frac{2EI}{6} = \frac{EI}{3}$$

$$k_{31} = 0$$



Column 2nd:

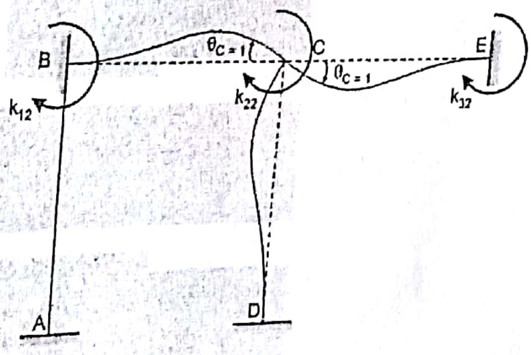
Give unit displacement in the direction of coordinate (2).

$\therefore\ \theta_C = 1$ and ensure $\theta_B = \theta_E = 0$

$$k_{12} = k_{21} = \frac{EI}{3}$$

$$k_{22} = \frac{4EI}{6} + \frac{4EI}{6} + \frac{4EI}{6} = 2EI$$

$$k_{32} = \frac{2EI}{6} = \frac{EI}{3}$$



Consider 3rd:

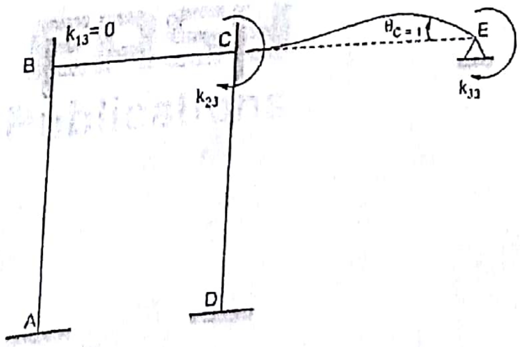
Give unit displacement in the direction of coordinate (3).

$\therefore\ \theta_E = 1$ and ensure $\theta_B = \theta_C = 0$

$$k_{13} = k_{13} = 0$$

$$k_{23} = k_{32} = \frac{EI}{3}$$

$$k_{33} = \frac{4EI}{6} = \frac{2EI}{3}$$



The stiffness matrix for selected coordinate is,

$$[k] = \begin{bmatrix} \frac{4EI}{3} & \frac{EI}{3} & 0 \\ \frac{EI}{3} & 2EI & \frac{EI}{3} \\ 0 & \frac{EI}{3} & \frac{2EI}{3} \end{bmatrix}$$

$$[k] = \frac{EI}{3}\begin{bmatrix} 4 & 1 & 0 \\ 1 & 6 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

The inverse of stiffness matrix $[k]$

$$\begin{array}{c|cccc} 4 & 1 & 0 & 4 & 1 \\ \hline 1 & 6 & 0 & 1 & 6 \\ 0 & 1 & 2 & 0 & 1 \\ 4 & 1 & 0 & 4 & 1 \\ 1 & 6 & 1 & 1 & 6 \end{array}$$

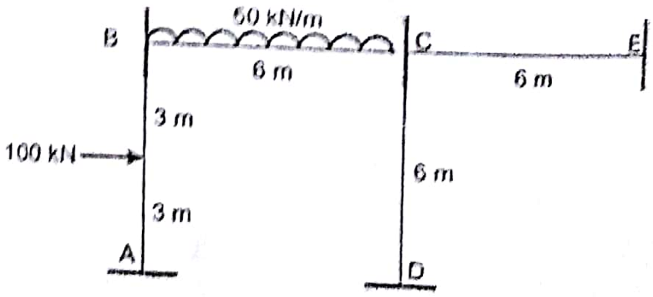
$$Adj\,[k] = \frac{3}{EI}\begin{bmatrix} 11 & -2 & 1 \\ -2 & 8 & -4 \\ 1 & -4 & 23 \end{bmatrix}$$

$$|k| = 4(12-1) - 1(2-0) + 0$$
$$|k| = 42$$

$$[k]^{-1} = \frac{Adj\,[k]}{|k|} = \frac{3}{EI}\begin{bmatrix} 0.261 & -0.00476 & 0.0238 \\ -0.00476 & 0.1904 & -0.0952 \\ 0.0238 & -0.0952 & 0.5476 \end{bmatrix}$$

$$[k]^{-1} = \frac{1}{EI}\begin{bmatrix} 0.783 & -0.142 & 0.0714 \\ -0.142 & 0.5712 & -0.2856 \\ 0.0714 & -0.2856 & 1.6428 \end{bmatrix}$$

Let P_{1L}, P_{2L} and P_{3L} are forces developed in coordinate direction due to the given load in locked structure



$$P_{1L} = \bar{M}_{BA} - \bar{M}_{BC}$$
$$= \frac{100\times 6}{8} - \frac{50\times 6^2}{12}$$
$$= -75 \text{ kN-m}$$
$$P_{2L} = 150 \text{ kN-m}$$
$$P_{3L} = 0$$

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{bmatrix} = \frac{1}{EI}\begin{bmatrix} 0.783 & -0.142 & 0.0714 \\ -0.142 & 0.5712 & -0.2856 \\ 0.0714 & -0.2856 & 1.6428 \end{bmatrix}\begin{bmatrix} -(-75) \\ -(150) \\ -0 \end{bmatrix}$$

$$\Delta_1 = \frac{1}{EI}[0.783\times 75 + 0.142\times 50 + 0]$$

$$\Rightarrow \quad \theta_B = \Delta_1 = \frac{80.02}{EI}$$

and $$\Delta_2 = \frac{1}{EI}[-0.142\times 75 - 0.5712\times 150 + 0]$$

$$\Rightarrow \quad \theta_C = \Delta_2 = -\frac{96.33}{EI}$$

and
$$\Delta_3 = \frac{1}{EI}[0.0714 \times 75 + 0.2856 \times 150 + 0]$$

$$\theta_E = \Delta_3 = \frac{48.19}{EI}$$

Final end moments:

$$M_{AB} = \overline{M}_{AB} + \frac{2EI}{6}\left[2\theta_A + \theta_B - \frac{3\Delta}{L}\right]$$

$$= -75 + \frac{2EI}{6}\left[0 + \frac{80.02}{EI} - 0\right] = -48.33 \text{ kNm}$$

$$M_{BA} = \overline{M}_{BA} + \frac{2EI}{6}\left[\theta_A + 2\theta_B - \frac{3\Delta}{L}\right]$$

$$= 75 + \frac{2EI}{6}\left[0 + 2 \times \frac{80.02}{EI} - 0\right] = 128.35 \text{ kNm}$$

$$M_{BC} = \overline{M}_{BC} + \frac{2EI}{6}\left[2\theta_B + \theta_C - \frac{3\Delta}{L}\right]$$

$$= -150 + \frac{2EI}{6}\left[\frac{80.02 \times 2}{EI} - \frac{96.33}{EI} - 0\right] = -128.76 \text{ kNm}$$

$$M_{CB} = \overline{M}_{CB} + \frac{2EI}{6}\left[\theta_B + 2\theta_C - \frac{3\Delta}{L}\right]$$

$$= 150 + \frac{2EI}{6}\left[\frac{80.02}{EI} - \frac{2 \times 96.33}{EI} - 0\right] = 112.47 \text{ kNm}$$

$$M_{DC} = \overline{M}_{DC} + \frac{2EI}{6}\left[2\theta_D + \theta_C - \frac{3\Delta}{L}\right]$$

$$= 0 + \frac{2EI}{6}\left[0 - \frac{96.33}{EI} - 0\right] = -32.11 \text{ kNm}$$

$$M_{CD} = \overline{M}_{CD} + \frac{2EI}{6}\left[\theta_D + 2\theta_C - \frac{3\Delta}{L}\right]$$

$$= 0 + \frac{2EI}{6}\left[0 + \frac{2 \times -96.33}{EI} - 0\right] = -64.22 \text{ kNm}$$

$$M_{EC} = \overline{M}_{EC} + \frac{2EI}{6}\left[2\theta_E + \theta_C - \frac{3\Delta}{L}\right]$$

$$= 0 + \frac{2EI}{6}\left[\frac{2 \times 48.19}{EI} - \frac{96.33}{EI} - 0\right] = 0 \text{ kNm}$$

$$M_{CE} = \overline{M}_{CE} + \frac{2EI}{6}\left[\theta_E + 2\theta_C - \frac{3\Delta}{L}\right]$$

$$= 0 + \frac{2EI}{6}\left[\frac{48.19}{EI} - \frac{2 \times 96.33}{EI}\right] = -48.14 \text{ kNm}$$

Illustrative Examples

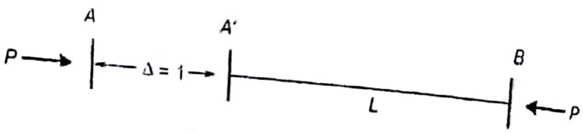
Example 10.19 Develop the stiffness matrix for the end loaded prismatic beam element AB with reference to the coordinates shown in figure.

Solution:

First column: To develop the first column of stiffness matrix, give unit displacement in the direction of coordinate 1 and find force induce in other coordinate directions

$$\Delta = \frac{PL}{AE}$$

if $\Delta = 1$ then $P = k$



$$k_{11} = \frac{AE}{L}$$

and

$$k_{21} = -\frac{AE}{L}$$

By Maxwell's reciprocal theorem,

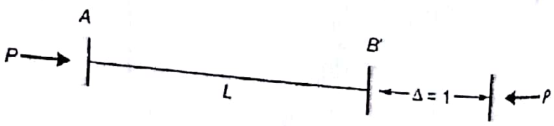
$$k_{12} = k_{21} = -\frac{AE}{L}$$

$k_{31} = 0,\ k_{41} = 0,\ k_{51} = 0,\ k_{61} = 0$

Second Column:

$$k_{22} = \frac{AE}{L}$$

$$k_{12} = -\frac{AE}{L}$$



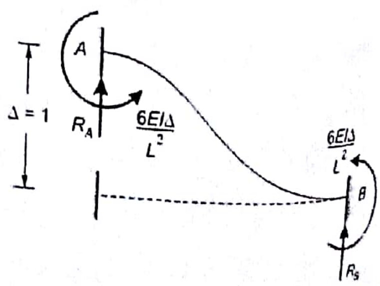
$k_{32} = 0,\ k_{42} = 0,\ k_{52} = 0,\ k_{62} = 0$

Third Column:

$k_{13} = 0\ \ k_{23} = 0$

$$k_{53} = -\frac{6EI}{L^2}$$

$$k_{63} = -\frac{6EI}{L^2}$$



$$\Sigma M_B = 0 \qquad R_A \times L - \frac{12EI}{L^2} = 0$$

$$R_A = \frac{12EI}{L^3}$$

and

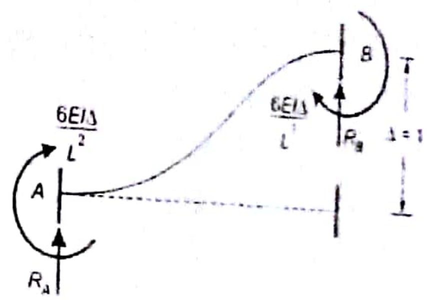
$$R_B = -\frac{12EI}{L^3}$$

$$k_{33} = \frac{12EI}{L^3}$$

$$k_{43} = -\frac{12EI}{L^3}$$

Fourth Column:

$\Sigma M_B = 0$, $\quad R_A \times L + \frac{12EI}{L^2} = 0$



$$R_A = -\frac{12EI}{L^3}$$

and

$$R_B = \frac{12EI}{L^3}$$

$$k_{44} = \frac{12EI}{L^3}$$

$$k_{34} = -\frac{12EI}{L^3}$$

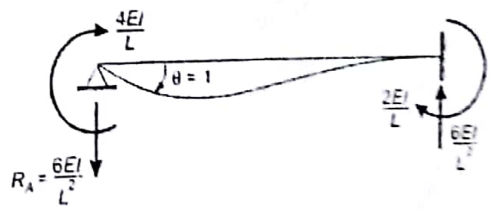
$k_{14} = 0$, $k_{24} = 0$

$$k_{54} = \frac{6EI}{L^2}$$

$$k_{64} = \frac{6EI}{L^2}$$

Fifth Column:

$$k_{55} = \frac{4EI}{L}$$



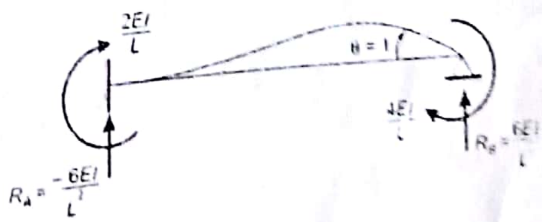
$k_{15} = 0$, $k_{25} = 0$

$$k_{35} = -\frac{6EI}{L^2}$$

$$k_{45} = \frac{6EI}{L^2}$$

$$k_{65} = \frac{2EI}{L}$$

Sixth Column:



$$k_{66} = \frac{4EI}{L}$$

$$k_{56} = \frac{2EI}{L}$$

$k_{16} = 0$, $k_{26} = 0$

$$k_{36} = -\frac{6EI}{L}$$

$$k_{46} = \frac{6EI}{L}$$